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A q-generalization of Laplace transforms

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Abstract. The Laplace transform is generalized by using the q-exponential function $e_q^x \equiv [1 + (1-q)x]^{1/(1-q)}$ that emerges from Tsallis' non-extensive statistical mechanics, and some of its properties are obtained. The usual transform is recovered as a limiting case ($q \to 1$). The use of the q-Laplace transform is illustrated by establishing a relation between the classical canonical q-partition function and the density of states.

1. Introduction

Among the integral transforms, Laplace's occupies a special place, mainly because of its usefulness in solving differential equations of functions of exponential order with initial value conditions or semi-infinite boundary value conditions. It has applications in various areas of science and engineering. A particular use of the Laplace transform within Boltzmann–Gibbs extensive statistical mechanics is to establish the connection between the density of states (an entirely mechanical property) and the canonical partition function.

There is an increasing focus on non-extensive phenomena in the physics literature and particularly on the Tsallis generalization of statistical mechanics. Since its formulation [1, 2], the theoretical body of the formalism has expanded significantly (see [3] for a recent and broad review). It has been applied to a variety of systems, among which we mention the Lévy [4] and correlated [5] anomalous diffusion, self-gravitating systems [6], peculiar velocities of galaxies [7], turbulence in pure electron plasma [8], solar neutrinos [9] and quantum scattering of spinless particles [10].

The present work is included in the formal developments of mathematical methods associated with Tsallis statistical mechanics. Some previous works along these lines are on distribution functions [11], linear response theory [12], perturbative and variational methods [13], Green's functions [14], path integral and Bloch equations [15], consistent testing [16] and trigonometric and hyperbolic functions [17].

The starting point of the mathematical developments associated with the Tsallis formalism is the definition of the generalized q-logarithm and q-exponential functions [17, 18]

$$\ln_q x \equiv \frac{x^{1-q} - 1}{1-q} \qquad \exp_q x \equiv \mathrm{e}_q^x \equiv [1 + (1-q)x]^{1/(1-q)}. \tag{1}$$

These functions are a kind of q-deformation of the usual ones and are reduced to them in the limit $q \to 1$. Their definitions allow one to write a sharp analogy between Boltzmann–Gibbs statistical mechanics and Tsallis generalization. For instance, the generalized entropy of the microcanonical ensemble is written as $S_q = k \ln_q W$ ($k \in \mathcal{R} > 0$ and W is the number of microstates).

In the Tsallis non-extensive statistical mechanics, there is a generalized q-partition function Z_q. We show that the density of states may be recovered from Z_q by an inverse q-Laplace transform.

2. q-Laplace transform

In order to obtain a generalization of the Laplace transform of a function f,

$$\mathcal{L}\{f(t)\}(s) \equiv F(s) \equiv \int_0^\infty f(t) \exp_1(-st)\,\mathrm{d}t \tag{2}$$

motivated by non-extensive Tsallis ideas, we consider the replacement of $\exp_1(-st)$ by a q-exponential. We can achieve this by the following simple possibilities: replace the kernel $\exp_1(-st)$ by

(a) $\exp_q(-st)$,
(b) $[\exp_q(-t)]^s$, or
(c) $[\exp_q(+t)]^{-s}$.

All of these possibilities reduce to the usual kernel e_1^{-st} in the limit $q \to 1$. In the present work we consider the second case and define the q-Laplace transform of a function f by

$$\mathcal{L}_q\{f(t)\}(s) \equiv F_q(s) \equiv \int_0^\infty f(t)[\exp_q(-t)]^s\,\mathrm{d}t. \tag{3}$$

We shall show that this particular generalization has a variety of interesting properties; the other two possible generalizations will be commented on later. This definition has the usual Laplace transform as a particular case, when $q \to 1$. For $q < 1$, we must use a *cut-off*, essentially the same as that used in the non-extensive statistical mechanics: the q-density matrix for the canonical ensemble of a system with Hamiltonian $\hat{H}$ is given by [2]

$$\hat{\rho}_q = \frac{1}{Z_q}\left[1 - (1-q)\beta\hat{H}\right]^{1/(1-q)} \tag{4}$$

with

$$Z_q = \mathrm{Tr}\left[1 - (1-q)\beta\hat{H}\right]^{1/(1-q)}. \tag{5}$$

In order to retain a consistent probabilistic interpretation (eigenvalues of $\hat{\rho}_q$ must be non-negative real numbers monotonically decreasing with the energy) a cut-off condition is introduced, which imposes $\rho(E_n) = 0$ whenever $\left[1 - (1-q)\beta\hat{H}\right] \leqslant 0$ ($\{E_n\}$ is the set of eigenvalues of the Hamiltonian $\hat{H}$). To be coherent with the non-extensive formalism, we also adopt the cut-off condition: $\exp_q(-t) \equiv 0$ whenever $[1-(1-q)t] \leqslant 0$.

We shall begin with the following definition: a function $f(t)$ defined on the interval $a \leqslant t < \infty$ is said to be of *q-exponential order* σ_0 ($\sigma_0 \in \mathcal{R}$) if there exists $M \in \mathcal{R}$ such that $\left|[\exp_q(-t)]^{\sigma_0} f(t)\right| \leqslant M$. To demonstrate the existence of the q-Laplace transform, let $f(t)$ be measurable and of q-exponential order σ_0. Then, following Lebesgue's dominated convergence theorem, we have $\left|f(t)[\exp_q(-t)]^s\right| \leqslant g(t)$ where

$$g(t) = \begin{cases} [1-(1-q)t]^{-(\sigma-\sigma_0)/(q-1)} & q > 1 \\ [1-(1-q)t]^{-(\sigma-\sigma_0)/(q-1)}\chi_{t\leqslant(1-q)^{-1}} & q < 1. \end{cases} \tag{6}$$

The function $g(t)$ is obviously integrable. Thus, the integral $\int_0^\infty \mathrm{d}t\ f(t)[\exp_q(-t)]^s$ converges for $\mathrm{Re}(s) > \sigma_0 + (q-1)$.

The inverse of the q-Laplace transform is given by

$$\mathcal{L}_q^{-1}\{F_q(s)\}(t) = f(t) = \frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} F_q(s)[\exp_q(-t)]^{-s-(1-q)}\,\mathrm{d}s \tag{7}$$

where c is a real constant that exceeds the real part of all the singularities of $F_q(s)$. The proof is found by checking the identities

$$f(t) = \mathcal{L}_q^{-1}\{\mathcal{L}_q\{f(t)\}\} \tag{8}$$

and

$$F_q(s) = \mathcal{L}_q\{\mathcal{L}_q^{-1}\{F_q(s)\}\}. \tag{9}$$

The first identity is proved as follows:

$$\begin{aligned}
\mathcal{L}_q^{-1}\{\mathcal{L}_q\{f(t)\}\} &= \frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \mathcal{L}_q\{f(t)\}[1-(1-q)t]^{-s/(1-q)-1}\,\mathrm{d}s \\
&= \frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \left[\int_0^\infty f(t')\big[1-(1-q)t'\big]^{s/(1-q)}\,\mathrm{d}t'\right] \\
&\quad \times [1-(1-q)t]^{-s/(1-q)-1}\,\mathrm{d}s \\
&= \int_0^\infty \frac{f(t')}{[1-(1-q)t]} \left\{\frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \left[\frac{1-(1-q)t'}{1-(1-q)t}\right]^{s/(1-q)} \mathrm{d}s\right\} \mathrm{d}t' \\
&= \int_0^\infty \frac{f(t')}{[1-(1-q)t]} \\
&\quad \times \left\{\frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \exp_1\!\left(s \ln_1\left[\frac{1-(1-q)t'}{1-(1-q)t}\right]^{1/(1-q)}\right) \mathrm{d}s\right\} \mathrm{d}t'.
\end{aligned} \tag{10}$$

If we take into account the representation of the Dirac δ-function

$$\delta(x) = \frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \exp_1(\alpha x)\,\mathrm{d}\alpha \tag{11}$$

and also the property of a function $f(x)$ with a single, simple root at x_0

$$\delta(f(x)) = \frac{1}{|\mathrm{d}f/\mathrm{d}x|_{x=x_0}}\delta(x-x_0) \tag{12}$$

we can find equation (8) straightforwardly.

We can check equation (9) by defining

$$g(t) = \frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} F_q(z)[\exp_q(-t)]^{-z-(1-q)}\,\mathrm{d}z \tag{13}$$

where $F_q(s) = \mathcal{L}_q\{g(t)\}$ and c is such that the above integral converges. The q-Laplace transform of $g(t)$ is

$$\mathcal{L}_q\{g(t)\} = \frac{1}{2\pi \mathrm{i}} \int_0^\infty \mathrm{d}t\,[\exp_q(-t)]^s \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \mathrm{d}z\,F_q(z)[\exp_q(-t)]^{-z-(1-q)}. \tag{14}$$

Interchanging the order of the integrals (uniform convergence required), we have

$$\mathcal{L}_q\{g(t)\} = \frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \mathrm{d}z\,F_q(z) \int_0^\infty \mathrm{d}t\,[\exp_q(-t)]^{s-z-(1-q)}. \tag{15}$$

We require $\mathrm{Re}(z) = c < \mathrm{Re}(s)$ in order to guarantee the convergence of the second integral. We find, then

$$\mathcal{L}_q\{g(t)\} = \frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \mathrm{d}z\, \frac{F_q(z)}{s-z}. \tag{16}$$

In order to evaluate this integral, we choose a contour defined by the straight line $\mathrm{Re}(z) = c$ and an arc to the right such that the pole s is located inside it. If $F_q(z)$ has no singularities to the right of $\mathrm{Re}(z) = c$, is of order $\mathrm{O}(z^{-k})$ (i.e. $|F_q(z)| < M|z|^k$ as $|z| \to \infty$, $M, k \in \mathcal{R} > 0$) in this half-plane, and the integral over the arc gives no contribution, then, by the Cauchy integral formula, we find that $g(t)$ and $f(t)$ possess the same Laplace transform $F_q(s)$.

3. Properties of the q-Laplace transform

In the following we list some properties of the present q-Laplace transform (their proofs are formally simple and thus are not included).

(a) Limiting values

$$\lim_{s\to\infty} s\mathcal{L}_q\{f(t)\} = \lim_{t\to 0} f(t) \tag{17}$$

$$\lim_{s\to 0} s\mathcal{L}_q\{f(t)\} = \lim_{t\to\infty} \{[1-(1-q)t]f(t)\}. \tag{18}$$

(b) Linearity

$$\mathcal{L}_q\{a_1 f_1(t) + a_2 f_2(t)\} = a_1\mathcal{L}_q\{f_1(t)\} + a_2\mathcal{L}_q\{f_2(t)\}. \tag{19}$$

(c) Scaling

$$\mathcal{L}_q\{f(at)\} = \frac{1}{a}F_{q'}(s/a) \qquad \text{with} \quad q' = 1-(1-q)/a. \tag{20}$$

(d) Attenuation, or substitution

$$F_q(s-s_0) = \mathcal{L}_q\left\{[\exp_q(-t)]^{-s_0} f(t)\right\}. \tag{21}$$

(e) q-shifting, or q-translation

$$\mathcal{L}_q\left\{ f\left(\frac{t-t_0}{1-(1-q)t_0}\right)\theta\left(\frac{t-t_0}{1-(1-q)t_0}\right)\right\} = [\exp_q(-t_0)]^{s-(1-q)} F_q(s) \tag{22}$$

where $\theta(t)$ is the Heaviside step function.

(f) Transform of derivatives

We may express these properties in two forms:

$$\mathcal{L}_q\{f'(t)\} = s\mathcal{L}_q\left\{\frac{f(t)}{1-(1-q)t}\right\} - f(0) \tag{23}$$

$$\mathcal{L}_q\{f''(t)\} = s(s-(1-q))\mathcal{L}_q\left\{\frac{f(t)}{[1-(1-q)t]^2}\right\} - f'(0) - sf(0) \tag{24}$$

and

$$\mathcal{L}_q\left\{\frac{\mathrm{d}}{\mathrm{d}t}[[1-(1-q)t]f(t)]\right\} = s\mathcal{L}_q\{f(t)\} - f(0) \qquad \text{for} \quad s > q-1 \tag{25}$$

$$\mathcal{L}_q\left\{\frac{\mathrm{d}}{\mathrm{d}t}\left[[1-(1-q)t]\frac{\mathrm{d}}{\mathrm{d}t}[[1-(1-q)t]f(t)]\right]\right\}$$
$$= s^2\mathcal{L}_q\{f(t)\} - f'(0) - sf(0) + (1-q)f(0). \tag{26}$$

The most common application of the Laplace transform is in the solution of linear differential equations. It takes advantage of the property $\mathcal{L}_1\{f(t)\} = s\mathcal{L}_1\{f(t)\} - f(0)$, to transform differential equations into algebraic equations in the s domain. In this q-generalized version, the corresponding properties (equations (23) and (25)) may also be used, with the same purpose, for solving differential equations in which the derivatives appear in the form of $(\mathrm{d}/\mathrm{d}t)\{[1-(1-q)t]f(t)\}$. In particular, $\exp_q(\pm\lambda t)$, $\lambda > 0$ (the q-exponential emerges in a variety of physical situations within the non-extensive statistical mechanics) is a solution of the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\{[1 \pm (1-q)\lambda t]f(t)\} = \pm(2-q)\lambda f(t). \tag{27}$$

(g) Derivative of transforms

$$F_q'(s) = \mathcal{L}_q\{\ln_1[\exp_q(-t)]f(t)\} \tag{28}$$

$$F_q^{(n)}(s) = \mathcal{L}_q\{\ln_1^n[\exp_q(-t)]f(t)\}. \tag{29}$$

(h) Transform of integrals
We have here two possible forms

$$\mathcal{L}_q\left\{\int_0^t f(\lambda)\,\mathrm{d}\lambda\right\} = \frac{1}{s+1-q}\mathcal{L}_q\{[1-(1-q)t]f(t)\} \tag{30}$$

and

$$\mathcal{L}_q\left\{\frac{\int_0^t f(\lambda)\,\mathrm{d}\lambda}{1-(1-q)t}\right\} = \frac{1}{s}\mathcal{L}_q\{f(t)\}. \tag{31}$$

(i) Integration of transforms

$$\int_s^\infty F_q(u)\,\mathrm{d}u = \mathcal{L}_q\left\{\frac{-f(t)}{\ln_1[\exp_q(-t)]}\right\} \tag{32}$$

$$\int_s^\infty \cdots \int_s^\infty F_q(u)\,\mathrm{d}^n u = \mathcal{L}_q\left\{\frac{(-1)^n f(t)}{\ln_1^n[\exp_q(-t)]}\right\}. \tag{33}$$

(j) Product of transforms

$$\mathcal{L}_q\{f(t)\}\mathcal{L}_q\{g(t)\} = \mathcal{L}_q\{f(t) *_q g(t)\} \tag{34}$$

where $(f *_q g)(t)$ is the q-convolution product, defined by [19]

$$\begin{aligned}(f *_q g)(t) &\equiv \int_0^t \mathrm{d}\lambda \int_0^\lambda \mathrm{d}\lambda' f(\lambda)g(\lambda')\delta(t - [\lambda + \lambda' - (1-q)\lambda\lambda'])\\ &= \int_0^t f\left(\frac{t-\lambda}{1-(1-q)\lambda}\right)\frac{g(\lambda)}{1-(1-q)\lambda}\,\mathrm{d}\lambda.\end{aligned} \tag{35}$$

In fact, transformation (35) is a straightforward extension of the *parallel product* introduced in [20]. The q-convolution is commutative ($f *_q g = g *_q f$), distributive with respect to addition and multiplication ($f *_q (ag + bh) = a(f *_q g) + b(f *_q h)$), where a and b are constants) and associative ($f *_q (g *_q h) = (f *_q g) *_q h$).

4. q-Laplace transforms of some elementary functions

Next we list the q-Laplace transforms of some particular functions:

(a) Unit function, Dirac δ-function and Heaviside step function:

$$\mathcal{L}_q\{1\} = \frac{1}{s+1-q} \qquad \begin{cases} s > q-1 & \text{for} \quad q \geqslant 1 \\ s > 0 & \text{for} \quad q \leqslant 1 \end{cases} \tag{36}$$

$$\mathcal{L}_q\{\delta(t)\} = 1 \tag{37}$$

$$\mathcal{L}_q\{\theta(t-t_0)\} = \frac{[\exp_q(-t_0)]^{s+1-q}}{s+1-q} \qquad \begin{cases} s > q-1 & \text{for} \quad q \geqslant 1 \\ s > 0 & \text{for} \quad q \leqslant 1. \end{cases} \tag{38}$$

(b) Power functions: for integer powers, we have

$$\begin{aligned} \mathcal{L}_q\{t^{n-1}\} &= \frac{(n-1)!}{[s+(1-q)][s+2(1-q)]\cdots[s+n(1-q)]} \\ &= \frac{(n-1)!}{s^n Q_n(2-q')} \end{aligned} \tag{39}$$

with $n = 1, 2, 3\ldots$, $s > n(q-1)$ for $q > 1$, $s > 0$ for $q < 1$ and $(1-q) = s(1-q')$. $Q_n(q)$ is a polynomial function given by [17]

$$Q_n(q) \equiv 1 \cdot q(2q-1)(3q-2)\cdots[nq-(n-1)]. \tag{40}$$

For real (not necessarily integer) powers, we make use of the Hilhorst integral representation of $\exp_q(-x)$ for $q > 1$ $(x > 0)$ [21]

$$\exp_q(-x) = \frac{1}{\Gamma(1/(q-1))} \int_0^{\infty} u^{1/(q-1)-1} \mathrm{e}_1^{-u} \mathrm{e}_1^{-(q-1)xu}\, \mathrm{d}u \tag{41}$$

and the integral representation for $q < 1$ $(x > 0)$ [22]

$$\exp_q(-x) = \frac{\Gamma((2-q)/(1-q))}{2\pi} \int_{-\infty}^{+\infty} \frac{\mathrm{e}_1^{1+\mathrm{i}u}}{(1+\mathrm{i}u)^{(2-q)/(1-q)}} \mathrm{e}_1^{-(1-q)(1+\mathrm{i}u)x}\, \mathrm{d}u \tag{42}$$

which brings implicitly the cut-off, and find

$$\mathcal{L}_q\{t^{\alpha-1}\} = \begin{cases} \Gamma(\alpha)\dfrac{\Gamma(s/(q-1)-\alpha)}{(q-1)^{\alpha}\Gamma(s/(q-1))} & s > \alpha(q-1) \qquad \text{for} \quad q \geqslant 1 \\ \Gamma(\alpha)\dfrac{\Gamma(s/(1-q)+1)}{(1-q)^{\alpha}\Gamma(s/(1-q)+\alpha+1)} & s > 0 \qquad \text{for} \quad q \leqslant 1. \end{cases} \tag{43}$$

(c) Exponential, circular and hyperbolic functions

The function e_1^{-at} $(a > 0)$ is of q-exponential order $\forall q$, and e_1^{at} is of q-exponential order for $q < 1$ (due to the cut-off). Their q-Laplace transforms are (see equations 3.383 5. and 3.383 1. of [23]):

$$\mathcal{L}_{q>1}\{\mathrm{e}_1^{-at}\} = \frac{1}{q-1}\Psi\left(1, 2-\frac{s}{q-1}; \frac{a}{q-1}\right) \tag{44}$$

$$\mathcal{L}_{q<1}\{\mathrm{e}_1^{\pm at}\} = \frac{1}{s+1-q}\,{}_1F_1\left(1, \frac{s}{1-q}+2; \frac{\pm a}{1-q}\right) \tag{45}$$

where $\Psi(\alpha, \gamma; z)$ and ${}_1F_1(\alpha, \gamma; z)$ are the confluent hypergeometric functions. For the circular and hyperbolic functions, for $q < 1$, we have

$$\mathcal{L}_{q<1}\{\sin(at)\} = -\frac{\mathrm{i}}{2}\frac{1}{s+1-q} \times \left[{}_1F_1\left(1, \frac{s}{1-q}+2; \frac{\mathrm{i}a}{1-q}\right) - {}_1F_1\left(1, \frac{s}{1-q}+2; \frac{-\mathrm{i}a}{1-q}\right)\right] \tag{46}$$

$$\mathcal{L}_{q<1}\{\cos(at)\} = \frac{1}{2}\frac{1}{s+1-q} \times \left[{}_1F_1\left(1, \frac{s}{1-q}+2; \frac{\mathrm{i}a}{1-q}\right) + {}_1F_1\left(1, \frac{s}{1-q}+2; \frac{-\mathrm{i}a}{1-q}\right)\right] \tag{47}$$

$$\mathcal{L}_{q<1}\{\sinh(at)\} = \frac{1}{2}\frac{1}{s+1-q} \times \left[{}_1F_1\left(1, \frac{s}{1-q}+2; \frac{a}{1-q}\right) - {}_1F_1\left(1, \frac{s}{1-q}+2; \frac{-a}{1-q}\right)\right] \tag{48}$$

$$\mathcal{L}_{q<1}\{\cosh(at)\} = \frac{1}{2}\frac{1}{s+1-q} \times \left[{}_1F_1\left(1, \frac{s}{1-q}+2; \frac{a}{1-q}\right) + {}_1F_1\left(1, \frac{s}{1-q}+2; \frac{-a}{1-q}\right)\right]. \tag{49}$$

(d) q-exponential function
The function $f(t) = \mathrm{e}_{q'}^{at}$ with $q' = 1 + (1-q)/a$ is of q-exponential order a and its q-Laplace transform is

$$\mathcal{L}_q\{\mathrm{e}_{q'}^{at}\} = \frac{1}{s+1-q-a} \qquad \begin{cases} s > a+q-1 & \text{for} \quad q > 1 \\ s > 0 & \text{for} \quad q < 1. \end{cases} \tag{50}$$

We have also the following relations (see equations 3.197 3. and 3.197 5. of [23]):

$$\mathcal{L}_{q>2}\{\mathrm{e}_q^{at}\} = \frac{1}{a(q-2)}\,{}_2F_1\left(\frac{s}{q-1}, 1; 2-\frac{1}{q-1}; -a^{-1}\right) \qquad \begin{cases} s > 0 \\ a > 1 \end{cases} \tag{51}$$

$$\mathcal{L}_{q>1}\{\mathrm{e}_q^{-at}\} = \frac{1}{s+2-q}\,{}_2F_1\left(\frac{1}{q-1}, 1; \frac{s+1}{q-1}; 1-a\right) \qquad \begin{cases} s > q-2 \\ 0 < a < 2 \end{cases} \tag{52}$$

$$\mathcal{L}_{q<1}\{\mathrm{e}_q^{\pm at}\} = \frac{1}{s+1-q}\,{}_2F_1\left(\frac{-1}{1-q}, 1; \frac{s}{1-q}+2; \mp a\right) \qquad \begin{cases} s > 0 \\ |a| < 1 \end{cases} \tag{53}$$

$$\mathcal{L}_{q<1}\{\mathrm{e}_q^{-at}\} = \frac{1}{a(2-q)}\,{}_2F_1\left(\frac{-s}{1-q}, 1; \frac{1}{1-q}+2; a^{-1}\right) \qquad \begin{cases} s > 0 \\ a > 1 \end{cases} \tag{54}$$

where ${}_2F_1(\alpha, \beta; \gamma; z)$ is the Gaussian hypergeometric function. From equations (51)–(54) we find the q-Laplace transforms of the q-hyperbolic sine and cosine functions [17]

$$\sinh_q x = \tfrac{1}{2}\left(\mathrm{e}_q^{x} - \mathrm{e}_q^{-x}\right) \qquad \cosh_q x = \tfrac{1}{2}\left(\mathrm{e}_q^{x} + \mathrm{e}_q^{-x}\right). \tag{55}$$

5. Density of states and the classical q-partition function

To conclude this work we shall use the q-Laplace transform to establish a relation between the classical q-partition function and the density of states. We first use the *unnormalized* q-expectation value as defined in [2] (here with a continuous distribution of probabilities $\rho(\boldsymbol{r})$ where $\boldsymbol{r}$ is a dimensionless variable in the phase space)

$$\langle O\rangle_q = \int [\rho(\boldsymbol{r})]^q \mathrm{O}(\boldsymbol{r})\,\mathrm{d}\boldsymbol{r}. \tag{56}$$

Later on we shall focus on the so-called *normalized* q-expectation value. The q-partition function Z_q which emerges from the optimization of the generalized entropy [1]

$$S_q \equiv k\frac{1-\int \mathrm{d}\boldsymbol{r}\,[\rho(\boldsymbol{r})]^q}{q-1} \tag{57}$$

with the constraint $\langle\mathcal{H}\rangle_q = \text{constant}$ ($\mathcal{H}$ is the Hamiltonian) and the usual norm constraint

$$\int \rho(\boldsymbol{r})\,\mathrm{d}\boldsymbol{r} = 1 \tag{58}$$

is

$$Z_q(\beta) = \int \exp_q[-\beta\mathcal{H}(\boldsymbol{r})]\,\mathrm{d}\boldsymbol{r} \tag{59}$$

which may be rewritten as

$$Z_q(\beta) = \int_0^\infty g(E)\exp_q(-\beta E)\,\mathrm{d}E \tag{60}$$

where $g(E)$ is the density of states (i.e. $g(E)\,\mathrm{d}E$ is the number of states with energies lying between E and $E + \mathrm{d}E$). Now we make the change of variables $\epsilon = \beta E$ and introduce a dummy parameter η in equation (60) in order to identify it with equation (3),

$$\begin{aligned} Z_q(\beta) = Z_q(\beta,\eta)\big|_{\eta=1} &= \frac{1}{\beta}\int_0^\infty g(\epsilon/\beta)[1-(1-q)\epsilon]^{\eta/(1-q)}\,\mathrm{d}\epsilon\bigg|_{\eta=1} \\ &= \frac{1}{\beta}\mathcal{L}_q\{g(\epsilon/\beta)\}(\eta)\bigg|_{\eta=1}. \end{aligned} \tag{61}$$

According to equation (7), its inverse is given by

$$\begin{aligned} g(E) &= \frac{1}{2\pi\mathrm{i}}\int_{c-i\infty}^{c+\mathrm{i}\infty} Z_q(\beta,\eta)[1-(1-q)\epsilon]^{-\eta/(1-q)-1}\,\mathrm{d}\eta\bigg|_{\epsilon=\beta E} \\ &= \mathcal{L}_q^{-1}\{Z_q(\beta,\eta)\}(\epsilon)\big|_{\epsilon=\beta E}. \end{aligned} \tag{62}$$

Equation (61) may be used to find the q-partition function once a density of states is given, and equation (62) may be used in the reverse procedure. Let us illustrate this point with the classical ideal gas, whose q-partition function should be rewritten as

$$Z_q(\beta) = \frac{1}{N!}\int\prod_i \frac{\mathrm{d}^3x_i\,\mathrm{d}^3p_i}{h^3}\left[\exp_q\left(-\beta\sum_j\frac{p_j^2}{2m}\right)\right]^\eta\Bigg|_{\eta=1}. \tag{63}$$

The q-partition function (63) for the case $q < 1$ becomes [24]

$$Z_{q<1}(\beta) = \frac{V^N}{N!h^{3N}}\left(\frac{2\pi m}{(1-q)\beta}\right)^{3N/2}\frac{\Gamma(\eta/(1-q)+1)}{\Gamma\big(\eta/(1-q)+\frac{3}{2}N+1\big)}\Bigg|_{\eta=1}. \tag{64}$$

For the case $q > 1$, we have [21, 25]

$$Z_{q>1}(\beta) = \frac{V^N}{N!h^{3N}} \left(\frac{2\pi m}{(q-1)\beta} \right)^{3N/2} \frac{\Gamma\left(\eta/(q-1) - \frac{3}{2}N\right)}{\Gamma(\eta/(q-1))} \bigg|_{\eta=1}. \tag{65}$$

The integration of equation (62) (see equations (20) and (22), pp 349–50 of [26]) in both cases yields

$$g(E) = \frac{V^N}{N!h^{3N}} \frac{(2\pi m)^{3N/2}}{\Gamma\left(\frac{3}{2}N\right)} E^{3N/2-1} \tag{66}$$

which is the density of states of the classical ideal gas [27]. In order to have a q-Laplace transform, the density of states must be of q-exponential order. In the case where $q < 1$, the cut-off guarantees the admissibility condition, but in the case of $q > 1$, $g(E)$ is of q-exponential order (and, thus, admits a q-Laplace transform, and therefore a q-partition function) only if $1 < q < 1 + 2/(3N)$ (for large N). This range of validity is the same found by [21, 25] and says, as a consequence, that there is no classical ideal gas with $q > 1$ in the thermodynamic limit ($N \to \infty$).

Now we use the *normalized* q-expectation value, introduced in [28]

$$\begin{aligned} \langle\langle O \rangle\rangle_q &= \frac{\int [\rho(\boldsymbol{r})]^q \mathrm{O}(\boldsymbol{r})\, \mathrm{d}\boldsymbol{r}}{\int [\rho(\boldsymbol{r})]^q\, \mathrm{d}\boldsymbol{r}} \\ &= \frac{\langle O \rangle_q}{\langle 1 \rangle_q}. \end{aligned} \tag{67}$$

The q-partition function which follows from the optimization of (57) with the constraints (58) and $\langle\langle \mathcal{H} \rangle\rangle_q = U_q$, where U_q is the (constant) q-generalized internal energy, is

$$\overline{Z}_q(\beta) = \int \exp_q\left[-\beta \frac{(\mathcal{H}(\boldsymbol{r}) - U_q)}{\int [\rho(\boldsymbol{r}')]^q\, \mathrm{d}\boldsymbol{r}'} \right] \mathrm{d}\boldsymbol{r} \tag{68}$$

$$= \exp_q\left[\frac{\beta U_q}{\int [\rho(\boldsymbol{r})]^q\, \mathrm{d}\boldsymbol{r}} \right] Z'_q(\beta') \tag{69}$$

where β is the Lagrange parameter, β' is defined by

$$\beta' \equiv \frac{\beta}{\int [\rho(\boldsymbol{r})]^q\, \mathrm{d}\boldsymbol{r} + (1-q)\beta U_q} \tag{70}$$

and $Z'_q(\beta')$ has the same functional form as the unnormalized q-partition function (60)

$$Z'_q(\beta') = \int_0^\infty g(E) \exp_q(-\beta' E)\, \mathrm{d}E. \tag{71}$$

With the change of variables $\epsilon = \beta' E$ we have

$$Z'_q(\beta') = Z'_q(\beta', \eta)\big|_{\eta=1} = \frac{1}{\beta'} \mathcal{L}_q\left\{ g(\epsilon/\beta') \right\}(\eta) \bigg|_{\eta=1} \tag{72}$$

and

$$g(E) = \mathcal{L}_q^{-1}\left\{ Z'_q(\beta', \eta) \right\}(\epsilon)\big|_{\epsilon=\beta' E}. \tag{73}$$

We finally address the other possible kernels for defining the generalization of the Laplace transform, suggested in the beginning of section 2. The third one consists of using the kernel $[\exp_q(+t)]^{-s}$. In this case, the cut-off would be introduced for $q > 1$ and $t \geqslant 1/(q-1)$. The case where $q < 1$ would have no cut-off. A similar procedure was used in [29] in another context. This possibility is entirely equivalent to ours with the change of variables $q = 2 - q'$ in equation (3), and essentially brings nothing new. Our choice has the advantage of placing the cut-off consistently with Tsallis formalism. The use of the kernel $\exp_q(-st)$ (first possibility) is a different generalization and would link $Z_q(\beta)$ and $g(E)$ by a q-Laplace transform without needing a dummy parameter. The main difficulty of this possibility is, of course, to find its inverse. Such a development would be very welcome.

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References

[1] Tsallis C 1988 *J. Stat. Phys.* **52** 479–87
An updated bibliography may be found at the web page http://tsallis.cat.cbpf.br/biblio.htm
[2] Curado E M F and Tsallis C 1991 *J. Phys. A: Math. Gen.* **24** L69–72
Curado E M F and Tsallis C 1991 *J. Phys. A: Math. Gen.* **24** 3187 (corrigendum)
Curado E M F and Tsallis C 1992 *J. Phys. A: Math. Gen.* **25** 1019 (corrigendum)
[3] Tsallis C 1999 *Braz. J. Phys.* **29** 1–35
[4] Alemany P A and Zanette D H 1994 *Phys. Rev.* E **49** R956–8
Tsallis C, Levy S V F, de Souza A M C and Maynard R 1995 *Phys. Rev. Lett.* **75** 3589–93
Tsallis C, Levy S V F, de Souza A M C and Maynard R 1996 *Phys. Rev. Lett.* **77** 5442 (erratum)
[5] Tsallis C and Bukman D J 1996 *Phys. Rev.* E **54** R2197–200
Compte A and Jou D 1996 *J. Phys. A: Math. Gen.* **29** 4321–9
Stariolo D A 1997 *Phys. Rev.* E **55** 4806–9
[6] Plastino A R and Plastino A 1993 *Phys. Lett.* A **174** 384–6
Hamity V H and Barraco D E 1996 *Phys. Rev. Lett.* **76** 4664–6
[7] Lavagno A, Kaniadakis G, Rego-Monteiro M, Quarati P and Tsallis C 1998 *Astrophys. Lett. Commun.* **35** 449–55
[8] Boghosian B M 1996 *Phys. Rev.* E **53** 4754–63
Anteneodo C and Tsallis C 1997 *J. Mol. Liq.* **71** 255–67
[9] Kaniadakis G, Lavagno A and Quarati P 1996 *Phys. Lett.* B **369** 308–12
Quarati P, Carbone A, Gervino G, Kaniadakis G, Lavagno A and Miraldi E 1997 *Nucl. Phys.* A **621** 345–8c
[10] Ion D B and Ion M L D 1998 *Phys. Rev. Lett.* **81** 5714–7
Ion M L D and Ion D B 1999 *Phys. Rev. Lett.* **83** 463–7
[11] Souza A M C and Tsallis C 1997 *Physica* A **236** 52–7
[12] Rajagopal A K 1996 *Phys. Rev. Lett.* **76** 3469–73
[13] Lenzi E K, Malacarne L C and Mendes R S 1998 *Phys. Rev. Lett.* **80** 218–21
[14] Rajagopal A K, Mendes R S and Lenzi E K 1998 *Phys. Rev. Lett.* **80** 3907–10
[15] Lenzi E K, Malacarne L C and Mendes R S 1999 Path integral approach to the nonextensive canonical density matrix *Physica* to appear
[16] Tsallis C 1998 *Phys. Rev.* E **58** 1442–5
[17] Borges E P 1998 *J. Phys. A: Math. Gen.* **31** 5281–8
[18] Tsallis C 1994 *Quimica Nova* **17** 468–71
[19] Tsallis C and Prato D Private communication
[20] Tsallis C 1981 *Kinam/Rev. Fis. (Mexico)* **3** 79
Tsallis C and Magalhães A C N 1996 *Phys. Rep.* **268** 305–430
[21] Tsallis C 1994 Extensive versus nonextensive physics *New Trends in Magnetism, Magnetic Materials and their Applications* ed J L Morán-López and J M Sanchez (New York: Plenum) pp 451–63
[22] Lenzi E K, Mendes R S and Rajagopal A K 1999 *Phys. Rev.* E **59** 1398–407
[23] Gradshteyn I S, Ryzhik I M and Jeffrey A (ed) 1994 *Table of Integrals, Series and Products* 5th edn (San Diego, CA: Academic)
[24] Prato D 1995 *Phys. Lett.* A **203** 165–8

[25] Plastino A R, Plastino A and Tsallis C 1994 *J. Phys. A: Math. Gen.* **27** 5707–14
[26] Erdélyi A (ed) 1954 *Tables of Integral Transforms (California Institute of Technology, Bateman Manuscript Project, vol I)* (New York: McGraw-Hill)
[27] Pathria R K 1972 *Statistical Mechanics* (Oxford: Pergamon)
[28] Tsallis C, Mendes R S and Plastino A R 1998 *Physica* A **261** 534–54
[29] Treumann R A 1998 *Phys. Rev.* E **57** 5150–3