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Mixed states having Poissonian statistics: how to distinguish them from coherent states?

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Abstract

We consider mixed states and coherent states with the same (Poissonian) statistics. By analyzing various properties connected with off-diagonal elements of the density matrix for both states, we find which of them are able (or not) to distinguish these states. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

Extensive study of quantum states of the radiation field has been developed since the pioneer work by Glauber [1], after the discovery of the laser in 1960. Various quantum effects in the light field have been investigated since 1977 with the detection of photon antibunching [2], constituting the first conspicuous proof of a quantum effect exhibited by the radiation field. Sub-Poissonian statistics [3], squeezing [4,5], oscillations in the photon-statistics and interference in the phase space [6] are other nonclassical effects observed in the quantized electromagnetic field.

Many states of a single-mode electromagnetic field, like number states $|n\rangle$ [7], phase states $|\phi_m\rangle$ [8–10], squeezed states $|z,\alpha\rangle$ [4,5] and so on, manifest some of these properties. On the other hand, coherent states, eigenstates of the annihilation operator $\hat{a}$, possess Poisson statistics and minimal uncertainty in the field quadratures; these

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states lie on the border between the classical and the quantum worlds. A way to establish whether a given state is classical or not is to find the corresponding Glauber quasi-distribution function $P(\alpha)$, the representation of its density operator in the coherent basis. Classical states have positive definite P functions while for nonclassical states the P functions assume negative values or are more singular than a delta function. A coherent state $|\alpha_0\rangle$, for which $P(\alpha)=\delta^{(2)}(\alpha-\alpha_0)$, and a thermal (chaotic) state, which has a Gaussian–Glauber distribution, are classical while number, phase and squeezed states are highly nonclassical. Actually, all pure states which are not coherent, are nonclassical [11].

In the last two decades, the study of nonclassical properties shown by several states of the quantized radiation field has become one of the main goals in quantum optics. More recently, comparison between two states possessing some identical properties has also become a topic of investigation as, for example, pure states having statistics of thermal states [12,13] and noncoherent pure states having statistics of coherent states [14]. The aim of the present work is to discuss the mixtures of number states having Poissonian statistics, which will be referred to as Poissonian mixed states (PMS), comparing their properties with those of coherent states (CS) for which the statistics is also Poissonian. The general point raised here is: *how to distinguish states having the same statistics*? Although we will restrict the analysis to the field of quantum optics, many aspects of the discussion can be applied to any system modeled by a liner quantum oscillator.

This paper is organized as follows: in Section 2, we introduce the PMS, discuss their generation and comment on their statistical properties. In Section 3, we determine the quadrature variances and the phase distribution as properties that distinguish between PMS and CS. The atomic population inversion and the atomic scattering due to atom–field interaction for both kinds of states are discussed in Section 4 while the phase-space representations are presented in Section 5. In Section 6, final remarks are made.

2. Definition and generation of PMS

A general mixture of number states of a single mode of the quantized radiation field is represented by a density matrix in the form

$$\hat{\rho}=\sum_{m=0}^{\infty} p_m |m\rangle\langle m| \,, \tag{2.1}$$

where the probability weights p_m must be normalized to satisfy $\sum_{m=0}^{\infty} p_m = 1$, which guarantee that $\hat{\rho}$ has unit trace. If one chooses the coefficients p_m such that

$$p_m = \mathrm{e}^{-\bar{n}}\frac{\bar{n}^m}{m!} \,, \tag{2.2}$$

where $\bar{n}$ is the mean photon number

$$\bar{n}=\sum_{m=0}^{\infty} m\, p_m \,, \tag{2.3}$$

then it follows immediately that this mixed state possesses Poissonian statistics, that is $\langle(\Delta\hat{n})^2\rangle = \bar{n}$, and thus such a mixed state has the same statistics as a CS [15]. In other words, all indicators of the nature of the statistics, like the Mandel factor $Q = (\langle(\Delta\hat{n})^2\rangle - \bar{n})/\bar{n}$ and the second-order correlation function for zero time delay $g^{(2)}(0) = Q + 1$, will take the coherent state values ($Q = 0$, $g^{(2)}(0) = 1$) and so it is impossible to distinguish statistically between a Poissonian superposition of all number states (a CS) and a mixture of all number states with Poissonian weights (a PMS).

A PMS can be generated in a simple way since it corresponds to a CS with random phase [16,17], that is, a state with density matrix given by

$$\hat{\rho} = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\theta |\alpha\rangle\langle\alpha| , \tag{2.4}$$

where $\alpha = |\alpha|\exp(\mathrm{i}\theta)$. In fact,

$$\begin{aligned}\langle m|\hat{\rho}|m'\rangle &= \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\theta\langle m|\alpha\rangle\langle\alpha|m'\rangle \\ &= \mathrm{e}^{-|\alpha|^2}\frac{|\alpha|^m|\alpha|^{m'}}{\sqrt{m!}\sqrt{m'!}}\frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\theta\, \mathrm{e}^{\mathrm{i}(m-m')\theta} = \mathrm{e}^{-|\alpha|^2}\frac{|\alpha|^{2m}}{m!}\,\delta_{m,m'} ,\end{aligned} \tag{2.5}$$

so that, if one fixes $|\alpha|^2 = \bar{n}$, Eq. (2.4) coincides with (2.1), that is, the density matrix of the PMS with mean number of photons $\bar{n}$ is

$$\hat{\rho}_{\mathrm{PMS}} = \mathrm{e}^{-\bar{n}}\sum_{m=0}^{\infty}\frac{\bar{n}^m}{m!}|m\rangle\langle m| = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\theta|\sqrt{\bar{n}}\mathrm{e}^{\mathrm{i}\theta}\rangle\langle\sqrt{\bar{n}}\mathrm{e}^{\mathrm{i}\theta}| . \tag{2.6}$$

A coherent state can be easily produced since all states of a single-mode radiation field generated by classical currents are coherent [1]; also, lasers operating well above the threshold create coherent light fields. A PMS can then be generated by losing control on the phase of the source of an initially coherent field and this can be realized as a traveling wave as well as a field confined inside a cavity.

It is easy to see that a PMS tends to the vacuum state $|0\rangle$ as $\bar{n}\to 0$. For $\bar{n}\neq 0$, one can indicate the degree of loss of purity, measuring the departure of the density matrix from the idempotent property, by calculating the trace of $\hat{\rho}^2_{\mathrm{PMS}}$.

$$\begin{aligned}\mathrm{D} = \mathrm{Tr}\,[\hat{\rho}^2_{\mathrm{PMS}}] &= \mathrm{Tr}\left[\sum_{m=0}^{\infty} p_m^2|m\rangle\langle m|\right] \\ &= \mathrm{e}^{-2\bar{n}}\sum_{m=0}^{\infty}\frac{\bar{n}^{2m}}{(m!)^2} = \mathrm{e}^{-2\bar{n}}I_0(2\bar{n}) ,\end{aligned} \tag{2.7}$$

where we have used the series representation of the modified Bessel function of first kind [18]

$$I_0(z) = \sum_{n=0}^{\infty}\frac{z^{2n}}{4^n(n!)^2} . \tag{2.8}$$

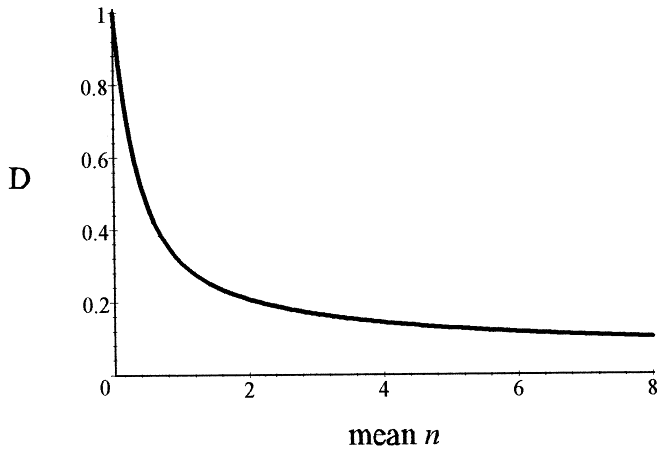

Fig. 1. Trace of the square of density matrix for a PMS, D, plotted as a function of the mean number of photons.

Fig. 1 presents a plot of D versus $\bar{n}$ showing that it decreases monotonically as $\bar{n}$ grows, tending to zero when $\bar{n} \to \infty$ as expected.

To distinguish a PMS, which is a field prepared from an incoherent mixture with a Poisson distribution, from a CS, one has either to perform phase-sensitive measurements, such as those involving quadratures variances, or to probe other properties depending on off-diagonal elements of $\hat{\rho}$ in the Fock's basis.

3. Quadrature variances and phase distribution

The quadrature operators corresponding to a field mode are defined by

$$\hat{x}_j = i^{1-j}(\hat{a} - (-1)^j \hat{a}^\dagger), \quad j = 1,2 \tag{3.1}$$

and therefore their commutator, in units of $\hbar$, is $[\hat{x}_1, \hat{x}_2] = 2i$. The mean values of these quadrature operators vanish for a PMS; in fact, using (2.6) one has

$$\langle \hat{x}_j \rangle_{\text{PMS}} = \text{Tr}(\hat{\rho}_{\text{PMS}} \hat{x}_j)$$

$$= \sum_{n,m=0}^{\infty} p_m \langle n | m \rangle \langle m | \hat{x}_j | n \rangle = \sum_{n=0}^{\infty} p_n \langle n | i^{1-j}(\hat{a} - (-1)^j \hat{a}^\dagger) | n \rangle = 0 \, . \tag{3.2}$$

On the other hand,

$$\langle \hat{x}_j^2 \rangle_{\text{PMS}} = \sum_{n=0}^{\infty} p_n \langle n | i^{2-2j} (\hat{a} - (-1)^j \hat{a}^\dagger)^2 | n \rangle = \sum_{n=0}^{\infty} p_n \langle n | (2\, \hat{a}^\dagger \hat{a} + 1) | n \rangle = 2\bar{n} + 1 \, , \tag{3.3}$$

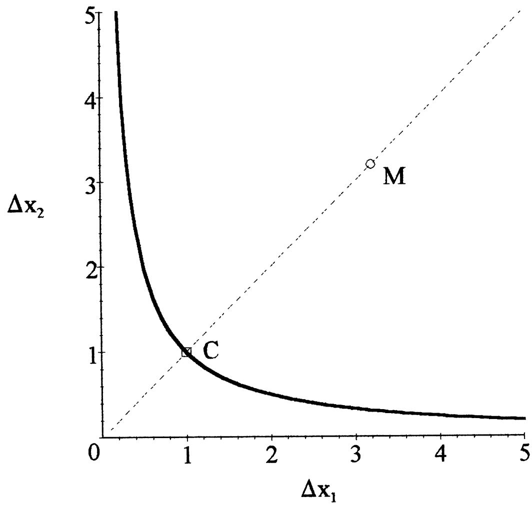

Fig. 2. Quadrature variances diagram: the point C represents all CS and M stands for a PMS; both coordinates of M are equal to $\sqrt{1+2\bar{n}}$. The hyperbola $\Delta x_1 \Delta x_2 = 1$ represents the Heisenberg relation boundary.

so that the quadrature variances of a PMS are given by

$$\langle (\Delta \hat{x}_j)^2 \rangle_{\text{PMS}} = 2\bar{n} + 1 \tag{3.4}$$

for both $j = 1, 2$.

One sees immediately that the above expression depends exclusively on the fact that the density matrix of a PMS is diagonal in the Fock's basis and therefore it holds for all mixtures of number states. In particular, the quadrature variances of a thermal state, for which the density matrix is given by (2.6) with [16]

$$p_m = \frac{1}{1+\bar{n}} \left(\frac{\bar{n}}{1+\bar{n}} \right)^m , \quad \bar{n} = (e^{\hbar\omega/kT} - 1)^{-1} , \tag{3.5}$$

where ω is the field frequency and T stands for the temperature of the radiation source, are identical to that of a PMS with the same value of $\bar{n}$ showing that the results (3.4) have nothing to do with the statistics. Notice also that if the mean number of photons is an integer, $\bar{n} = l$, then (3.4) coincides with the quadrature variances of the number state $|l\rangle$.

Although a PMS has the same (Poissonian) statistics as a CS, it is not a minimal uncertainty state; its quadrature noise enhances as its intensity is increased. In the diagram of quadrature variances shown in Fig. 2, the point C represents all CS while the point M represents a PMS, and also all mixtures of number states with mean number of photons $\bar{n}$; since $\bar{n}$ is a real number, M runs continuously from C along the diagonal line $\Delta x_1 = \Delta x_2 \geqslant 1$. In the limiting case where $M \to C$, that is $\bar{n} \to 0$, the PMS tends to the vacuo state.

One should also expect to distinguish a PMS from a CS by looking at their phase distributions, but to do so quantum-mechanically one has to define a phase observable.

One proposal is the Pegg–Barnett Hermitian phase operator, defined in the $(N+1)$-dimensional Hilbert space as [8–10]

$$\hat{\phi}_{\mathrm{PB}} = \sum_{m=0}^{N} \phi_m |\phi_m\rangle\langle\phi_m| \,, \tag{3.6}$$

where the kets

$$|\phi_m\rangle = \frac{1}{\sqrt{N+1}} \sum_{k=0}^{N} \exp(\mathrm{i}k\phi_m)|k\rangle \,, \tag{3.7}$$

with $\phi_m = \phi_0 + 2\pi m(N+1)^{-1}$, form a basis in this truncated space. In this case, it follows that

$$\langle\hat{\phi}_{\mathrm{PB}}\rangle_{\mathrm{N}} = \mathrm{Tr}(\hat{\rho}_{\mathrm{N}}\hat{\phi}_{\mathrm{PB}}) = \sum_{r,s=0}^{N} \langle r|\hat{\rho}_{\mathrm{N}}|s\rangle\langle s|\hat{\phi}_{\mathrm{PB}}|r\rangle = \sum_{m=0}^{N} \phi_m P_m^{(\mathrm{N})} \,, \tag{3.8}$$

where

$$P_m^{(\mathrm{N})} = \frac{1}{N+1} \sum_{r,s=0}^{N} \exp[\mathrm{i}(s-r)\phi_m]\, \langle r|\hat{\rho}_{\mathrm{N}}|s\rangle \,. \tag{3.9}$$

Notice that, for any mixture of number states, for which $\hat{\rho}_{\mathrm{N}} = \sum_{r=0}^{N} p_r |r\rangle\langle r|$, one finds $P_m^{(\mathrm{N})} = (N+1)^{-1} \sum_{r=0}^{N} p_r$. Now, defining the phase probability density as

$$P(\phi) = \lim_{N\to\infty} \left[\frac{N+1}{2\pi} P_m^{(\mathrm{N})}\right] \tag{3.10}$$

one sees that

$$\langle\hat{\phi}\rangle = \lim_{N\to\infty} \langle\hat{\phi}_{\mathrm{PB}}\rangle_{\mathrm{N}} \to \int_{\phi_0}^{\phi_0+2\pi} \mathrm{d}\phi\, \phi P(\phi) \,. \tag{3.11}$$

In the case of a PMS, and actually for all mixtures of number states, one obtains

$$P_{\mathrm{PMS}}(\phi) = \frac{1}{2\pi} \quad\Rightarrow\quad \langle\hat{\phi}\rangle_{\mathrm{PMS}} = \phi_0 + \pi \tag{3.12}$$

and $\langle(\Delta\hat{\phi})^2\rangle_{\mathrm{PMS}} = \pi^2/3$, which are the same results as those for a number state; as expected, all mixtures of number states have random phase. For a CS $|\alpha\rangle$ with $\alpha = |\alpha|\exp(i\theta)$, however, one finds, in the limit of large amplitude ($|\alpha| \gg 1$), $\langle\hat{\phi}\rangle_{\mathrm{CS}} = \theta$ and $\langle(\Delta\hat{\phi})^2\rangle_{\mathrm{CS}} = (2|\alpha|)^{-2}$ showing that, in this case, the coherent state has a well-defined phase [8–10].

4. Atomic population inversion and atomic scattering

Another way to distinguish a PMS from a CS possessing the same mean number of photons is to investigate properties which involve the interaction of the field with atoms since they also depend on off-diagonal elements of the density matrix of the field. One of these properties is the population inversion for a two-level (Rydberg) atom interacting with the field as described by the single-photon Jaynes–Cummings model in the rotating wave approximation,

$$\hat{H} = \frac{\hbar\omega_0}{2}\,\hat{\sigma}_3 + \hbar\omega\hat{a}^\dagger\hat{a} + \hbar\lambda(\hat{\sigma}_+\hat{a} + \hat{\sigma}_-\,\hat{a}^\dagger)\,, \tag{4.1}$$

where ω_0 is the frequency of the transition between the ground and the excited atomic states ($\hbar\omega_0 = E_{|e\rangle} - E_{|g\rangle}$), ω is the field-mode frequency and λ is the atom-field coupling parameter. In the above equation, $\hat{\sigma}_3$ is a Pauli matrix and $\hat{\sigma}_+$ and $\hat{\sigma}_-$ correspond to the raising and lowering operators in the atomic two-level basis, which are given by

$$\hat{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \hat{\sigma}_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \hat{\sigma}_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{4.2}$$

In this case, the time evolution operator, $\hat{U}(t,0) = \exp(-\mathrm{i}\hat{H}t/\hbar)$, has the following matrix representation in the atomic Hilbert space [19]:

$$\hat{U}(t,0) = \mathrm{e}^{-\mathrm{i}\omega\hat{n}t} \begin{pmatrix} \hat{O}_{11} & \hat{O}_{12} \\ \hat{O}_{21} & \hat{O}_{22} \end{pmatrix}, \tag{4.3}$$

where

$$\hat{O}_{11} = \mathrm{e}^{-\mathrm{i}\omega t/2}\left[\cos(\lambda t\sqrt{\hat{v}+1}) - \frac{i\delta}{2}\frac{\sin(\lambda t\sqrt{\hat{v}+1})}{\sqrt{\hat{v}+1}}\right], \tag{4.4}$$

$$\hat{O}_{12} = -i\,\mathrm{e}^{-\mathrm{i}\omega t/2}\,\frac{\hat{a}\sin(\lambda t\sqrt{\hat{v}})}{\sqrt{\hat{v}}}, \tag{4.5}$$

$$\hat{O}_{21} = -i\,\mathrm{e}^{\mathrm{i}\omega t/2}\,\frac{\hat{a}^\dagger\sin(\lambda t\sqrt{\hat{v}+1})}{\sqrt{\hat{v}+1}}, \tag{4.6}$$

$$\hat{O}_{22} = \mathrm{e}^{\mathrm{i}\omega t/2}\left[\cos(\lambda t\sqrt{\hat{v}}) + \frac{i\delta}{2}\frac{\sin(\lambda t\sqrt{\hat{v}})}{\sqrt{\hat{v}}}\right], \tag{4.7}$$

in which $\delta = (\omega_0 - \omega)\lambda^{-1}$ and $\hat{v} = \hat{n} + \delta^2/4$ with $\hat{n} = \hat{a}^\dagger\hat{a}$. For the atom–field system initially in a pure state $|\psi_{\mathrm{AF}}(0)\rangle$, one can determine $|\psi_{\mathrm{AF}}(t)\rangle = \hat{U}(t,0)|\psi_{\mathrm{AF}}(0)\rangle$ and then find the atomic inversion $W(t) = \langle\psi_{\mathrm{AF}}(t)|\hat{\sigma}_3|\psi_{\mathrm{AF}}(t)\rangle$, as done in Ref. [20]. To include the case of mixed states, one has to deal with the density matrices instead of wave functions.

Assume that the atom is initially in a normalized superposition of its ground and excited states, $|\psi_{\mathrm{A}}(0)\rangle = c_g|g\rangle + c_e|e\rangle$, and the field is described by the density matrix $\hat{\rho}_{\mathrm{F}}(0)$, so that $\hat{\rho}_{\mathrm{AF}}(0) = |\psi_{\mathrm{A}}(0)\rangle\langle\psi_{\mathrm{A}}(0)| \otimes \hat{\rho}_{\mathrm{F}}(0)$. At time t, the density matrix of the atom–field system is given by

$$\begin{aligned}\hat{\rho}_{\mathrm{AF}}(t) &= \hat{U}(t,0)\,\hat{\rho}_{\mathrm{AF}}(0)\,\hat{U}^\dagger(t,0) \\ &= \mathrm{e}^{-\mathrm{i}\omega\hat{n}t}\begin{pmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{pmatrix}\mathrm{e}^{\mathrm{i}\omega\hat{n}t}\,,\end{aligned} \tag{4.8}$$

where

$$\hat{A}_{11} = |c_e|^2\, \hat{O}_{11}\hat{\rho}_{\rm F}(0)\, \hat{O}^{\dagger}_{11} + c_e c_g^* \hat{O}_{11}\hat{\rho}_{\rm F}(0)\hat{O}^{\dagger}_{12} + c_e^* c_g \hat{O}_{12}\hat{\rho}_{\rm F}(0)\hat{O}^{\dagger}_{11} + |c_g|^2 \hat{O}_{12}\hat{\rho}_{\rm F}(0)\, \hat{O}^{\dagger}_{12}\,, \tag{4.9}$$

$$\hat{A}_{22} = |c_e|^2 \hat{O}_{21}\hat{\rho}_{\rm F}(0)\hat{O}^{\dagger}_{21} + c_e c_g^* \hat{O}_{21}\hat{\rho}_{\rm F}(0)\hat{O}^{\dagger}_{22} + c_e^* c_g \hat{O}_{22}\hat{\rho}_{\rm F}(0)\hat{O}^{\dagger}_{21} + |c_g|^2 \hat{O}_{22}\hat{\rho}_{\rm F}(0)\hat{O}^{\dagger}_{22}\,. \tag{4.10}$$

Similar expressions for $\hat{A}_{12}$ and $\hat{A}_{21}$ will not be presented, since they do not interfere in the calculation of averages of observables that are diagonal in the matrix representation in the atomic space. The atomic inversion then becomes

$$W(t) = {\rm Tr}[\hat{\rho}_{\rm AF}(t)\, \hat{\sigma}_3] = {\rm Tr}_{\rm F}\, [{\rm e}^{-{\rm i}\omega\hat{n}t}(\hat{A}_{11} - \hat{A}_{22}){\rm e}^{{\rm i}\omega\hat{n}t}] \tag{4.11}$$

which, in the Fock's basis, is given by (including detuning δ)

$$\begin{aligned} W_\delta(t) = {} & |c_e|^2 \left\{ 1 - 2\sum_{n=0}^{\infty} \frac{n+1}{v+1} \sin^2(\lambda t\sqrt{v+1}) \langle n|\hat{\rho}_{\rm F}(0)|n\rangle \right\} \\ & - |c_g|^2 \left\{ 1 - 2\sum_{n=0}^{\infty} \frac{n}{v} \sin^2(\lambda t\sqrt{v}) \langle n|\hat{\rho}_{\rm F}(0)|n\rangle \right\} \\ & + 2|c_e|\,|c_g| \sum_{n=0}^{\infty} \sqrt{n+1} \left[\sin(\phi+\gamma) \frac{\sin(2\lambda t\sqrt{v+1})}{\sqrt{v+1}} \right. \\ & \left. + \,\delta \cos(\phi+\gamma) \frac{\sin^2(\lambda t\sqrt{v+1})}{v+1} \right] |\langle n|\hat{\rho}_{\rm F}(0)|n+1\rangle|\,, \end{aligned} \tag{4.12}$$

where $v = n + \delta^2/4$, with ϕ and γ defined by $c_e\, c_g^* = |c_e|\, |c_g| \exp(-{\rm i}\phi)$ and $\langle n|\hat{\rho}_{\rm F}(0)|n+1\rangle = |\langle n|\hat{\rho}_{\rm F}(0)|n+1\rangle| \exp(-{\rm i}\gamma)$. For a pure state this expression reduces to that found by Chaba et al. [20]. For the resonant case, $\delta = 0$, Eq. (4.12) reduces to

$$\begin{aligned} W_0(t) = {} & |c_e|^2 \sum_{n=0}^{\infty} \cos(2\lambda t\sqrt{n+1}) \langle n|\hat{\rho}_{\rm F}(0)|n\rangle \\ & - |c_g|^2 \sum_{n=0}^{\infty} \cos(2\lambda t\sqrt{n}) \langle n|\hat{\rho}_{\rm F}(0)|n\rangle \\ & + 2|c_e|\,|c_g| \sum_{n=0}^{\infty} \sin(\phi+\gamma) \sin(2\lambda t\sqrt{n+1}) |\langle n|\hat{\rho}_{\rm F}(0)|n+1\rangle|\,. \end{aligned} \tag{4.13}$$

One sees immediately that the last term in the expression of $W_\delta(t)$ is identically null for a PMS since it involves off-diagonal elements of the density matrix, but survives for a coherent state in many circumstances. Naturally, if the atom starts in either the ground or the excited states then no distinction can be made between the atomic inversion due to interaction with the field in a PMS or in a coherent state with the same mean number of photons since, by construction, $\langle n|\hat{\rho}_{\rm PMS}(0)|n\rangle = \langle n|\hat{\rho}_{\rm CS}(0)|n\rangle = (\bar{n})^n \exp(-\bar{n})/n!$. However, if the initial atomic state is a superposition $c_g|g\rangle + c_e|e\rangle$ the field being in a coherent state $|\alpha\rangle$, with $\alpha = |\alpha|\exp(i\theta)$, one has $\gamma = \theta$, the atomic interference term does not vanish (except in the special case of zero detuning and $\phi + \theta = k\pi$) and the atomic population inversion distinguishes the PMS from the coherent state. This situation is illustrated in the figures below, where we consider some atomic superpositions and the field with frequency in resonance with the atomic transition. In Fig. 3 we have plotted the atomic inversion when the field is initially in the PMS, while in Fig. 4 the field starts in a coherent state with $\theta = \pi/2 - \phi$, a choice which maximizes the atomic interference term. One sees that the well-known collapses and revivals occurring in the case of a coherent field, are practically unaffected by the weights in the atomic superposition while for the PMS they are significantly depressed and coherent trapping occurs when one reaches an equally weighted superposition. The situation shown in Fig. 4 happens, for example, if the atomic superposition is in phase ($\phi = 0$) and the coherent state has $\alpha = iy_0$ or when α is real and the atomic state components are in quadrature ($\phi = \pi/2$). If the field frequency is detuned relative to the atomic transition then, no matter the value of $\phi + \theta$, the atomic inversion will be distinct for a PMS and a coherent state.

One can also distinguish a PMS from a CS, with the same $\bar{n}$, by looking at the scattering of two-level atoms by the field. As shown in Ref. [21], the scattering of Rydberg atoms through a node of a standing coherent field reveals a *nonsymmetric* momentum distribution of the atoms right after the interaction which yields the atomic endoscopy of the field state, since this allows one to reconstruct its number state representation. Recent studies [22] have shown that the atomic scattering by pure states possessing symmetric Wigner functions around the origin in phase space leads to a *symmetric* distribution of scattered momenta which does not allow atomic endoscopy. As it will be shown in the next section, the Wigner function of a PMS is symmetric in phase space and, therefore, the atomic scattering by a field in a PMS will produce a *symmetric* distribution of scattered atoms. As a consequence, by comparing the results for a PMS and a CS, the atomic scattering is able to distinguish these states.

5. Phase-space representations

A definitive way to distinguish a PMS from a CS with the same mean number of photons is to look at their representations in phase space, which is done below.

5.1. Glauber P function

The P representation, which is a diagonal representation in the coherent basis, is defined by

$$\hat{\rho} = \int d^2\beta\, P(\beta)|\beta\rangle\langle\beta| \,, \tag{5.1}$$

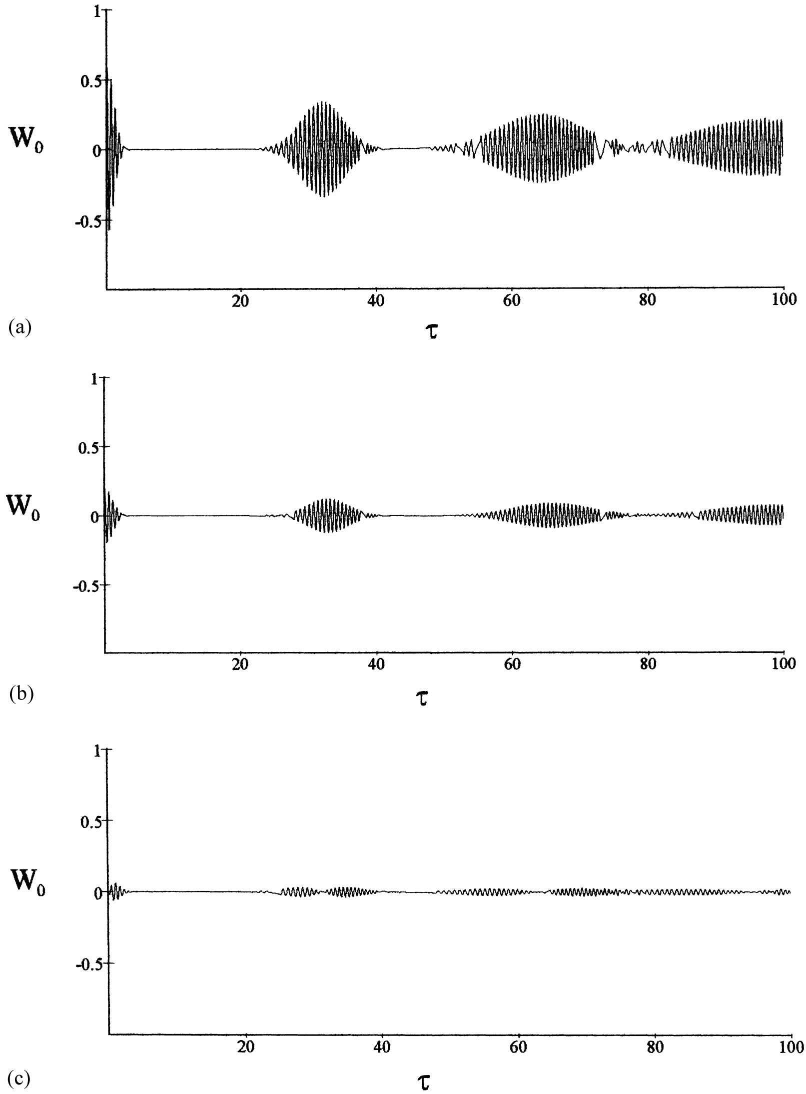

Fig. 3. Atomic population inversion for zero detuning, W_0, as a function of $\tau = \lambda t$ for a field in a PMS with $\bar{n} = 25$ and some atomic superpositions: (a) $|c_e^2| = 0.8$ and $|c_g^2| = 0.2$; (b) $|c_e^2| = 0.6$ and $|c_g^2| = 0.4$; (c) $|c_e^2| = |c_g^2| = 0.5$.

where $\beta = x + \mathrm{i}y$ and $\mathrm{d}^2\beta = \mathrm{d}x\,\mathrm{d}y$. It follows immediately that, for a coherent state $|\alpha\rangle$, the P function is given by

$$P_{\mathrm{CS}}(\beta) = \delta^{(2)}(\beta - \alpha)\,. \tag{5.2}$$

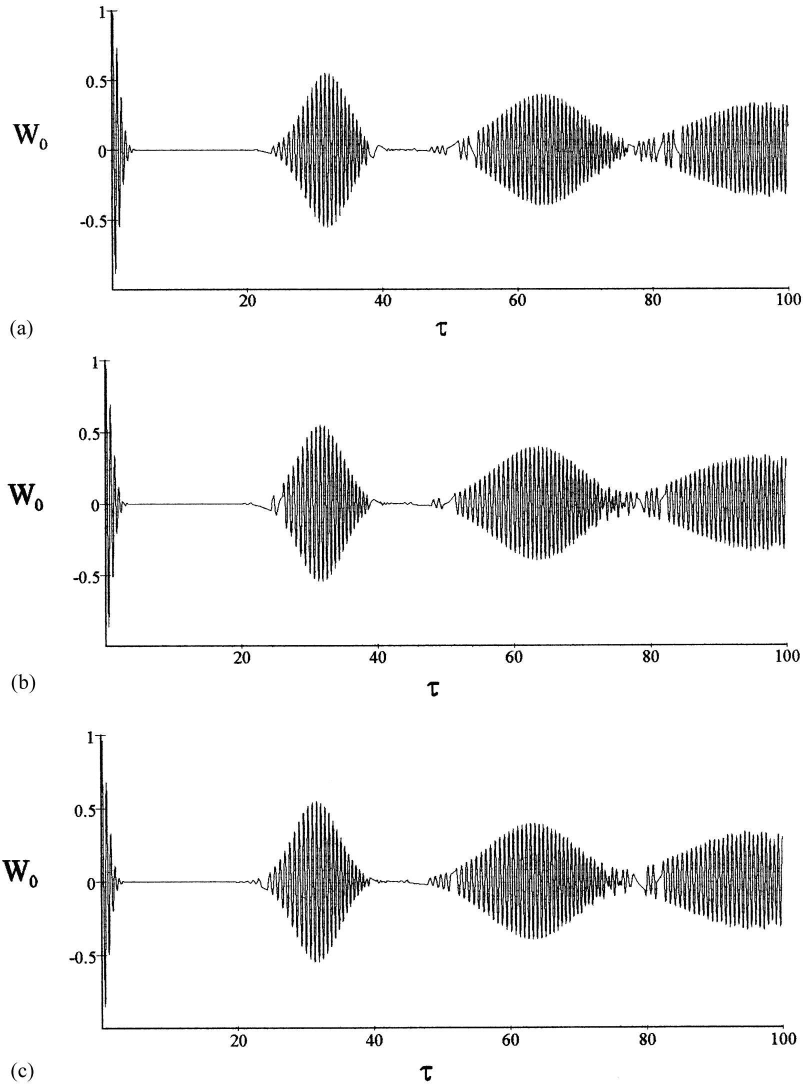

Fig. 4. Zero detuning atomic population inversion, W_0, as a function of $\tau = \lambda t$ for a field in a CS with $|\alpha^2| = 25$ and some atomic superpositions satisfying $\phi + \theta = \pi/2$: (a) $|c_e^2| = 0.8$ and $|c_g^2| = 0.2$; (b) $|c_e^2| = 0.6$ and $|c_g^2| = 0.4$; (c) $|c_e^2| = |c_g^2| = 0.5$.

Using this expression and the coherent basis representation of $\hat{\rho}_{\text{PMS}}$, one finds, for a PMS with mean number of photons $\bar{n}$ [16,17],

$$P_{\text{PMS}}(\beta) = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}\theta \, \delta^{(2)}(\beta - \sqrt{\bar{n}} \mathrm{e}^{\mathrm{i}\theta}) = \frac{1}{2\pi\sqrt{\bar{n}}} \delta(|\beta| - \sqrt{\bar{n}}) \,. \tag{5.3}$$

One sees that the singular nature of the P function is maintained but the distribution for the PMS becomes symmetric around the origin in the β plane.

5.2. Q function

The Q function is a regular and nonnegative representation defined as the diagonal matrix elements of the density operator in a coherent state

$$Q(\beta) = \frac{1}{\pi}\langle\beta|\hat{\rho}|\beta\rangle\,. \tag{5.4}$$

For example, the Q-function for a coherent state $|\alpha\rangle$, with $\alpha = x_0 + \mathrm{i}y_0$, is given by the well-known expression (with $\beta = x + \mathrm{i}y$)

$$Q_{\mathrm{CS}}(x,y) = \frac{1}{\pi}\exp[-(x-x_0)^2 - (y-y_0)^2]\,. \tag{5.5}$$

On the other hand, the Q-function of the PMS is given by

$$\begin{aligned} Q_{\mathrm{PMS}}(x,y) &= \frac{1}{\pi}\langle x+\mathrm{i}y|\hat{\rho}_{\mathrm{PMS}}|x+\mathrm{i}y\rangle \\ &= \frac{1}{\pi}\,\mathrm{e}^{-\bar{n}}\sum_{n=0}^{\infty}\frac{\bar{n}^n}{n!}|\langle x+\mathrm{i}y|n\rangle|^2 = \frac{1}{\pi}\,\mathrm{e}^{-\bar{n}-x^2-y^2}\sum_{n=0}^{\infty}\frac{\left[\sqrt{\bar{n}}\sqrt{x^2+y^2}\right]^{2n}}{(n!)^2} \\ &= \frac{1}{\pi}\exp(-x^2-y^2-\bar{n})I_0(2\sqrt{\bar{n}}\sqrt{x^2+y^2})\,, \end{aligned} \tag{5.6}$$

where we have used the representation (2.8) of I_0. This Q function can also be obtained from the coherent state expansion of $\hat{\rho}_{\mathrm{PMS}}$ given by (2.6). One verifies immediately that Q_{PMS}, which possesses a volcano shape, is symmetric around the origin in the xy-plane, as one should expect since the PMS corresponds to a coherent state with random phase, while Q_{CS} is a Gaussian bell centered at the point $\alpha_0 = x_0 + \mathrm{i}y_0$. These two Q-functions are illustrated in the Fig. 5, where we have taken $\bar{n} = |\alpha^2| = 5.0$.

5.3. Wigner function

The Wigner function, which may be defined as the Fourier transform of the symmetrically ordered characteristic function of the density matrix, can be expressed as

$$W(\beta) = \frac{1}{\pi^2}\int \mathrm{d}^2\eta\,\exp\left(-\frac{1}{2}\eta\eta^* + \eta^*\beta - \eta\beta^*\right)\,\mathrm{Tr}[\mathrm{e}^{-\eta^*\hat{a}}\hat{\rho}\,\mathrm{e}^{\eta\hat{a}^\dagger}]\,. \tag{5.7}$$

For a coherent state, one has $\mathrm{Tr}[\mathrm{e}^{-\eta^*\hat{a}}|\alpha\rangle\langle\alpha|\mathrm{e}^{\eta\hat{a}^\dagger}] = \exp[-\eta^*\alpha + \eta\alpha^*]$ and performing the η integration by completing the squares, one finds the known expression [15]

$$W_{\mathrm{CS}}(x,y) = \frac{2}{\pi}\exp[-2(x-x_0)^2 - 2(y-y_0)^2]\,. \tag{5.8}$$

In the case of a PMS, for which the density matrix is given by (2.6), one finds similarly

$$\begin{aligned} W_{\mathrm{PMS}}(x,y) &= \frac{1}{\pi^2}\int_0^{2\pi}\mathrm{d}\theta\,\exp[-2(\beta-\sqrt{\bar{n}}\mathrm{e}^{\mathrm{i}\theta})(\beta^*-\sqrt{\bar{n}}\mathrm{e}^{-\mathrm{i}\theta})] \\ &= \frac{1}{\pi^2}\exp[-2(x^2+y^2+\bar{n})]\int_0^{2\pi}\mathrm{d}\theta\,\exp[-4\sqrt{\bar{n}}(x\cos\theta + y\sin\theta)] \end{aligned}$$

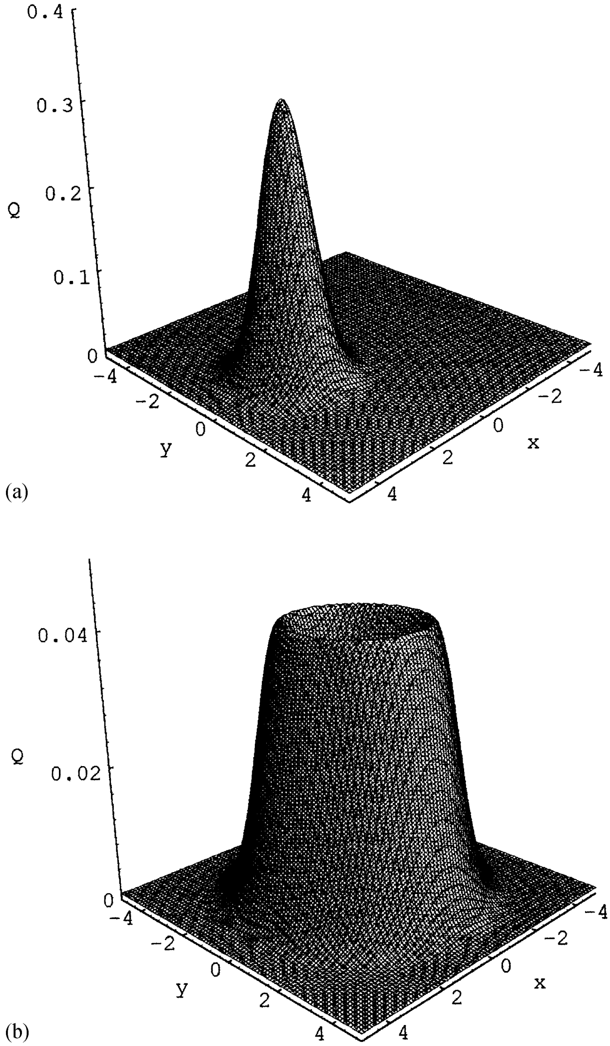

Fig. 5. Q function for (a) the coherent state $|\sqrt{5}\rangle$ and (b) for the Poissonian mixed state with $\bar{n} = 5$.

$$= \frac{2}{\pi} \exp[-2(x^2 + y^2 + \bar{n})]\, I_0(4\sqrt{\bar{n}}\sqrt{x^2 + y^2}) \,, \tag{5.9}$$

using the formula [14]

$$\int_0^{2\pi} d\theta \exp[-(q\cos\theta + p\sin\theta)] = 2\pi I_0(\sqrt{q^2 + p^2}) \,. \tag{5.10}$$

A comparison between the Wigner functions of a CS and a PMS is shown in Fig. 6. One should notice the similarities between the Wigner and the Q functions of the PMS: both have a volcano's shape but the Wigner function is sharper than the Q function, irrespective of the height scales. This is apparently a strange result since the Wigner

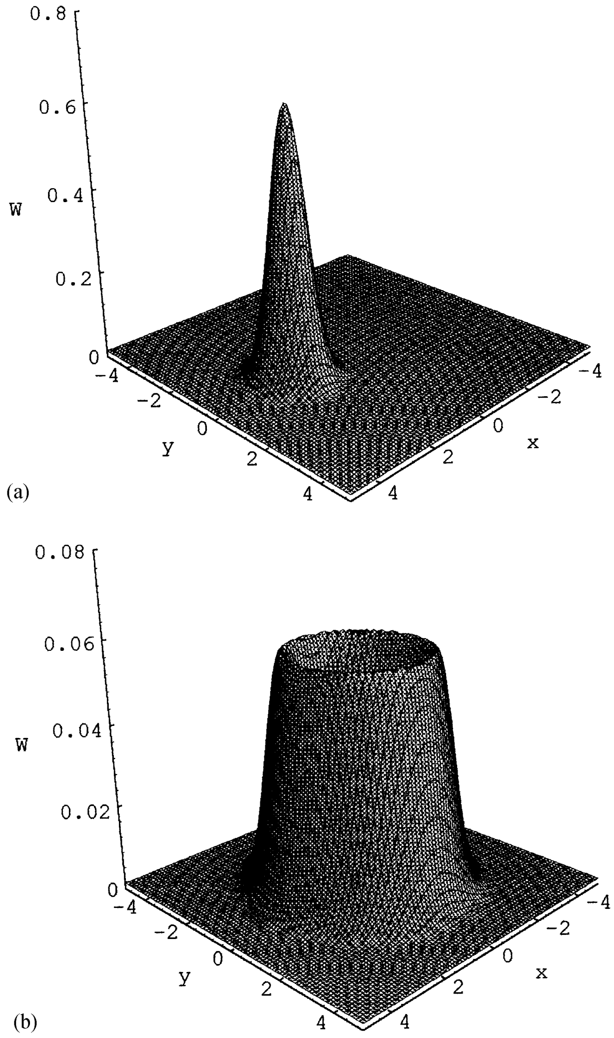

Fig. 6. Wigner function for (a) the CS $|\sqrt{5}\rangle$ and (b) for the PMS with $\bar{n} = 5$.

function may assume negative values whereas the Q function is always nonnegative. However, both are Gaussian convolutions of the P function

$$W(\beta) = \frac{2}{\pi}\int d^2\eta\, P(\eta)\, e^{-2|\eta-\beta|^2}, \quad Q(\beta) = \frac{1}{\pi}\int d^2\eta\, P(\eta) e^{-|\eta-\beta|^2}, \tag{5.11}$$

and, therefore, whenever the P function is a positive regular function or a delta function (which is the case for a PMS), the Wigner and the Q functions of the considered state will have similar shapes. This happens also for thermal state, for which these distributions are Gaussian, and holds in general for classical states of the field.

6. Conclusions

We have discussed the properties of Poissonian mixed states of a mode of the electromagnetic field comparing them with those of pure coherent states, which correspond to Poissonian superpositions of number states, with the same mean numbers of photons. Quadrature variances of a PMS increase linearly with its mean number of photons while all CS are minimal uncertainty states. So, quadrature dispersions better distinguish between a PMS and a CS when one has states of high excitation. This also happens with the field-phase distribution which is randomic for the PMS while the CS possesses a defined phase in the limit of large intensity. A clear cut distinction between a PMS and a CS is their phase-space representations because the P, Q and Wigner functions manifest explicitly the randomization of the phase of the PMS. Since a PMS and a CS (with same $\bar{n}$), by their definitions, possess the same statistics, they may also be distinguished by probing other properties that depend on off-diagonal elements of the density matrix in the Fock's basis. It was shown that the atomic population inversion and the scattering of Rydberg atoms interacting with the field provide a way to test whether the field is in a PMS or in a CS. These are some of the properties which might be investigated whenever one needs to verify if a source of coherent radiation has, by any means, become randomic in the phase of the field. Naturally, if the states (PMS or CS) concern with a traveling field, there are simpler ways to distinguish them, for example, by making a Young-interference experiment. Such a test, however, cannot be applied to stationary field states confined inside a high-Q cavity. In this case, the distinctive properties studied above become relevant.

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