

Bogoliubov transformation for quantum fields in $(S^1)^d \times \mathbb{R}^{D-d}$ topology and applications to the Casimir effect

This content has been downloaded from IOPscience. Please scroll down to see the full text.

2009 J. Phys.: Conf. Ser. 161 012024

(<http://iopscience.iop.org/1742-6596/161/1/012024>)

View [the table of contents for this issue](#), or go to the [journal homepage](#) for more

Download details:

IP Address: 200.128.60.106

This content was downloaded on 02/10/2013 at 14:28

Please note that [terms and conditions apply](#).

Bogoliubov transformation for quantum fields in $(\mathbb{S}^1)^d \times \mathbb{R}^{D-d}$ topology and applications to the Casimir effect

F C Khanna

Physics Department, Theoretical Physics Institute, University of Alberta, Edmonton, Alberta T6G 2J1 Canada

TRIUMF 4004, Westbrook mall, Vancouver, British Columbia V6T 2A3, Canada

E-mail: khanna@phys.ualberta.ca

J M C Malbouisson

Instituto de Física, Universidade Federal da Bahia, 40210-340, Salvador, Bahia, Brazil

E-mail: jmalboui@ufba.br

A E Santana

Instituto de Física, Universidade de Brasília, 70910-900, Brasília, DF, Brazil

E-mail: asantana@fis.unb.br

Abstract. A Bogoliubov transformation accounting simultaneously for spatial compactification and thermal effects is introduced. The fields are described in a $\Gamma_D^d = \mathbb{S}^{1_1} \times \dots \times \mathbb{S}^{1_d} \times \mathbb{R}^{D-d}$ topology, and the Bogoliubov transformation is derived by a generalization of the thermofield dynamics formalism, a real-time finite-temperature quantum field theory. We consider the Casimir effect for Maxwell and Dirac fields and for a non-interacting massless QCD at finite temperature. For the fermion sector in a cubic box, we analyze the temperature at which the Casimir pressure changes its sign from attractive to repulsive. This critical temperature is approximately 200 MeV when the edge of the cube is of the order of the confining lengths ($\approx 1 : fm$) for quarks in baryons.

Keywords: Thermofield dynamics, Bogoliubov transformation, Compactification, Casimir effect

1. Introduction

In the literature there are several studies considering the quantum field theory formulated in flat spaces, with non-trivial topologies. This is the case of space-time considered as a simply or non-simply connected D -dimensional manifold with topology of type $\Gamma_D^d = \mathbb{S}^{1_1} \times \dots \times \mathbb{S}^{1_d} \times \mathbb{R}^{D-d}$. The central characteristic of this kind of theory is that the topological structure of the space-time imposes modifications on the boundary conditions on fields and their Green functions, but it does not modify the local field equations. One known example of this kind of method is a quantum field at finite temperature, T , described by the Matsubara formalism. In this case the

thermal effect is a path-integral calculated on $\mathbb{S}^1 \times \mathbb{R}^{D-1}$, where \mathbb{S}^1 is a circumference of length $\beta = 1/T$ [1, 2]. As a consequence, the thermal field theory can be thought, in a generalized way, as a mechanism to deal simultaneously with spatial constraints and thermal effects in a field theory model. In this context, the $\lambda\phi^4$ theory, including considerations on the mass generation due to topological effects, was studied in Ref. [1, 3], while the temperature and topological aspects were analyzed in Ref. [2]. More recently, these ideas have been developed in the context of the Matsubara formalism [4, 5, 6, 7] as well as in thermofield dynamics (TFD); and applied to the Casimir effect considering the electromagnetic and fermion fields within a box [8, 9, 10]), to the $\lambda\phi^4$ model as the Ginsburg-Landau theory for superconductors [4, 5], and to the Gross-Neveu model [6, 7]. In the present paper, we apply this methodology to calculate the Casimir effect for a massless fermion field in a 3-dimensional box (a cube), by using a generalized TFD-Bogoliubov transformation, accounting for space compactification and the temperature effect.

An interest in the Casimir effect for fermions is the analysis of deconfinement in particle physics [11]–[24]. On the other hand, for this problem, the TFD-Bogoliubov transformation, as a real time formalism, gives rise to a useful theoretical ingredient, since the Green function is written naturally in two terms. One is the divergent free-space contribution; the other describes the compactification effects. This fact, not so direct in the imaginary time, allows one to treat the renormalization procedure in a convenient way [10].

TFD relies on two basic ingredients [25, 26, 27]. First, one defines a doubling of the original Fock space of the system leading to the expanded space $\mathcal{H}_T = \mathcal{H} \otimes \tilde{\mathcal{H}}$. This doubling is carried out by associating to each operator a acting on \mathcal{H} two operators in \mathcal{H}_T , A and \tilde{A} , which are connected by the tilde (dual) conjugation rules

$$\begin{aligned} (A_i A_j)^\sim &= \tilde{A}_i \tilde{A}_j, \\ (cA_i + A_j)^\sim &= c^* \tilde{A}_i + \tilde{A}_j, \\ (A_i^\dagger)^\sim &= (\tilde{A}_i)^\dagger, \\ (\tilde{A}_i)^\sim &= -\xi A_i, \end{aligned}$$

with $\xi = -1$ for bosons and $\xi = +1$ for fermions. The physical variables are described by non-tilde operators. The other basic ingredient of TFD is a Bogoliubov transformation, $B(\alpha)$, introducing a rotation in the tilde and non-tilde variables, in such a way that thermal effects emerge from a condensate state. In the standard doublet notation [27], we write

$$(A^r(\alpha)) = \begin{pmatrix} A(\alpha) \\ \xi \tilde{A}^\dagger(\alpha) \end{pmatrix} = B(\alpha) \begin{pmatrix} A \\ \xi \tilde{A}^\dagger \end{pmatrix}, \quad (1)$$

$(A^r(\alpha))^\dagger = (A^\dagger(\alpha), \tilde{A}(\alpha))$, with the Bogoliubov transformation given by

$$B(\alpha) = \begin{pmatrix} u(\alpha) & -v(\alpha) \\ \xi v(\alpha) & u(\alpha) \end{pmatrix}, \quad (2)$$

where $u^2(\alpha) + \xi v^2(\alpha) = 1$. The usual parametrization of the Bogoliubov transformation in TFD is obtained by setting $\alpha = \beta = T^{-1}$ and by requiring that $\langle 0, \tilde{0} | a^\dagger(\alpha) a(\alpha) | 0, \tilde{0} \rangle$ (with a^\dagger and a being the creation and the annihilation operators) gives either the Bose or the Fermi distribution, i.e.

$$u(\beta) = \left(1 + \xi e^{-\beta\varepsilon}\right)^{-\frac{1}{2}}, \quad v(\beta) = \left(e^{\beta\varepsilon} + \xi\right)^{-\frac{1}{2}}. \quad (3)$$

We consider the TFD approach for free fields aiming to extend the Bogoliubov transformation to account also for spatial compactification effects. The main application of our general discussion is the Casimir effect for cartesian confining geometries at finite temperature. For the sake of simplicity of notation, usually one identifies: $A \equiv a$ and $\tilde{A} \equiv \tilde{a}$.

2. Maxwell and massless Dirac fields in TFD

For a quantum field described by a Lagrangian density \mathcal{L} , the TFD Lagrangian density is given by $\mathcal{L}_T = \mathcal{L} - \tilde{\mathcal{L}}$, where $\tilde{\mathcal{L}}$ is the tilde conjugate of \mathcal{L} for the tilde fields. For the free-massless boson (Klein-Gordon) field, the two-point Green function in the doubled space is given by

$$\begin{aligned} G_0^{(ab)}(x - x') &= \langle 0, \tilde{0} | T [\phi(x)^a \phi(x')^b] | 0, \tilde{0} \rangle \\ &= \frac{-1}{(2\pi)^4} \int d^4 k G_0^{(ab)}(k) e^{-ik \cdot (x - x')}, \end{aligned} \quad (4)$$

where

$$G_0^{(ab)}(k) = \begin{pmatrix} G_0(k) & 0 \\ 0 & \xi G_0^*(k) \end{pmatrix} = \begin{pmatrix} \frac{1}{k^2 + i\epsilon} & 0 \\ 0 & \frac{-\xi}{k^2 - i\epsilon} \end{pmatrix}$$

(here $\xi = -1$, for we have bosons). In the configuration space, we have explicitly

$$G_0^{(1,1)}(x - x') = G_0(x - x') = \frac{-i}{(2\pi)^2} \frac{1}{(x - x')^2 - i\epsilon}.$$

The extension of this result for the electromagnetic field is simple. The free-photon propagator is written as

$$iD_{\mu\nu}^{(ab)}(x - x') = \langle 0, \tilde{0} | T [A_\mu^a(x) A_\nu^b(x')] | 0, \tilde{0} \rangle = g_{\mu\nu} G_0^{(ab)}(x - x').$$

For the free-massless Dirac field, the doubled Green function is given by

$$S_0^{(ab)}(x - x') = \begin{pmatrix} S_0(x - x') & 0 \\ 0 & -S_0^*(x' - x) \end{pmatrix},$$

where $S_0(x - x') = -i \langle 0 | T [\psi(x) \bar{\psi}(x')] | 0 \rangle = -i \gamma_\mu \partial^\mu G_0(x - x')$.

Now we use an α -dependent Bogoliubov transformation, performing a rotation among tilde and non-tilde variables. The α -dependent Green functions for the Klein-Gordon, Dirac and Maxwell fields are obtained through the Bogoliubov transformation given by Eq. (2) acting on $G_0^{(ab)}(k)$. (Notice that a Bogoliubov transformation has to be defined for each mode, k , of the field.) We have

$$G_0^{(ab)}(k; \alpha) = B_k^{-1(ac)}(\alpha) G_0^{(cd)}(k) B_k^{(db)}(\alpha); \quad (5)$$

explicitly, the components of $G_0^{(ab)}(k; \alpha)$ are given by

$$G_0^{11}(k; \alpha) = G_0(k) + \xi v_k^2(\alpha) [G_0^*(k) - G_0(k)], \quad (6)$$

$$G_0^{12}(k; \alpha) = G_0^{21}(k; \alpha) = \xi v_k(\alpha) u_k(\alpha) [G_0^*(k) - G_0(k)], \quad (7)$$

$$G_0^{22}(k; \alpha) = \xi G_0^*(k) + v_k^2(\alpha) [G_0(k) - G_0^*(k)]. \quad (8)$$

When $\xi = +1$, we obtain the auxiliary doubled two-point function which must be used for calculating the fermion propagator.

We are concerned with the α -dependent energy-momentum tensor, obtained from the usual expressions by replacing the fields by the Bogoliubov transformed counterparts. Actually, to treat thermal (and space confinement) effects, we will consider the renormalized vacuum expectation value of the α -dependent energy-momentum tensor defined, by subtracting the value corresponding to the free space at zero temperature, as

$$\mathcal{T}^{\mu\nu(ab)}(x; \alpha) = \langle 0, \tilde{0} | T^{\mu\nu(ab)}(x; \alpha) | 0, \tilde{0} \rangle - \langle 0, \tilde{0} | T^{\mu\nu(ab)}(x) | 0, \tilde{0} \rangle. \quad (9)$$

The physical results are obtained from the (11)-component. The doubled operators describing the energy-momentum tensor of free Maxwell and Dirac fields are given, respectively, by

$$T_M^{\mu\nu(ab)}(x; \alpha) = -F^{\mu\lambda(ab)}(x; \alpha)F_{\lambda}^{\nu(ab)}(x; \alpha) + \frac{1}{4}g^{\mu\nu}F_{\lambda\rho}^{(ab)}(x; \alpha)F^{\rho\lambda(ab)}(x; \alpha), \quad (10)$$

$$T_D^{\mu\nu(ab)}(x; \alpha) = \frac{i}{2}(\bar{\psi}^a(x; \alpha)\gamma^\mu\partial^\nu\psi^b(x; \alpha) - \partial^\nu\bar{\psi}^a(x; \alpha)\gamma^\mu\psi^b(x; \alpha)), \quad (11)$$

where $F_{\mu\nu}^{(ab)} = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a$ is the electromagnetic field tensor.

For accounting thermal effects, one takes $\alpha = \beta = T^{-1}$. In this case $v_k^2(\alpha)$ (given in Eq. (3) for one mode, with $k_0 = \varepsilon$) is written as

$$v_k^2(\beta) = \sum_{l=1}^{\infty} (-\xi)^{l+1} e^{-\beta k_0 l}, \quad (12)$$

so that Eq. (6) becomes

$$G_0^{11}(k; \beta) = G_0(k) + \sum_{l=1}^{\infty} (-\xi)^{l+1} e^{-\beta k_0 l} [G_0^*(k) - G_0(k)]. \quad (13)$$

In the space coordinate, we have

$$\begin{aligned} \bar{G}_0^{11}(x - x'; \beta) &= \sum_{l=1}^{\infty} (-\xi)^{l+1} [G_0^*(x' - x - i\beta l \hat{n}_0) \\ &\quad - G_0(x - x' - i\beta l \hat{n}_0)], \end{aligned} \quad (14)$$

where $\bar{G}_0^{11}(x - x'; \beta) = G_0^{11}(x - x'; \beta) - G_0(x - x')$ and $\hat{n}_0 = (1, 0, 0, 0)$ is a time-like vector. This expression is very useful for calculating $\mathcal{T}^{\mu\nu(11)}(x; \beta)$.

For the electromagnetic field, we have

$$\begin{aligned} \langle 0, \tilde{0} | T_M^{\mu\nu(ab)}(x; \beta) | 0, \tilde{0} \rangle &= -i \left\{ \Gamma^{\mu\nu}(x, x') G^{(ab)}(x - x'; \beta) \right. \\ &\quad \left. + 2 \left(\hat{n}_0^\mu \hat{n}_0^\nu - \frac{1}{4} g^{\mu\nu} \right) \delta(x - x') \delta^{ab} \right\} \Big|_{x \rightarrow x'}, \end{aligned}$$

where $\Gamma^{\mu\nu}(x, x') = 2(\partial^\mu \partial'^\nu - \frac{1}{4} g^{\mu\nu} \partial^\rho \partial'_\rho)$. This leads to

$$\begin{aligned} \mathcal{T}_M^{\mu\nu(11)}(\beta) &= -i \left\{ \Gamma^{\mu\nu}(x, x') \bar{G}_0^{11}(x - x'; \beta) \right\} \Big|_{x \rightarrow x'} \\ &= -\frac{2}{\pi^2} \sum_{l=1}^{\infty} \frac{g^{\mu\nu} - 4\hat{n}_0^\mu \hat{n}_0^\nu}{(\beta l)^4} = \frac{-\pi^2}{45\beta^4} (g^{\mu\nu} - 4\hat{n}_0^\mu \hat{n}_0^\nu), \end{aligned} \quad (15)$$

where we have used the Riemann *zeta*-function $\zeta(4) = \sum_{l=1}^{\infty} l^{-4} = \pi^4/90$. As expected, $E(T) = \mathcal{T}_M^{00(11)}(\beta) = \frac{1}{15} \pi^2 T^4$ gives the correct energy density of the photon gas at temperature T , the blackbody radiation, formulae.

For the free-massless fermion field, we obtain

$$\begin{aligned} \langle 0, \tilde{0} | T_D^{\mu\nu(ab)}(x; \beta) | 0, \tilde{0} \rangle &= \gamma^\mu \partial^\nu S^{(ab)}(x - x') \Big|_{x' \rightarrow x} \\ &= -4i \partial^\mu \partial^\nu G_0^{(ab)}(x - x') \Big|_{x' \rightarrow x}, \end{aligned}$$

leading to

$$\begin{aligned} \mathcal{T}_D^{\mu\nu(11)}(\beta) &= -4i\partial^\mu\partial^\nu[\bar{G}_0^{11}(x-x';\beta)]_{x'\rightarrow x} \\ &= \frac{4}{\pi^2} \sum_{l=1}^{\infty} (-1)^l \left[\frac{g^{\mu\nu} - 4\hat{n}_0^\mu\hat{n}_0^\nu}{(\beta l)^4} \right]. \end{aligned} \quad (16)$$

From this tensor, using $\sum_{l=1}^{\infty} (-1)^l l^{-4} = -7\pi^4/720$, we recover the well known result for the internal energy density of the Dirac field at temperature T , $E(T) = \mathcal{T}_D^{00(11)}(\beta) = \frac{7\pi^2}{60} T^4$. In the following section we introduce a more general Bogoliubov transformations taking into account not only temperature, but also the space compactification of a field.

3. Generalized Bogoliubov transformations

The preceding results show that the equilibrium TFD is equivalent to the Matsubara imaginary-time formalism, which has been used also to consider spatial compactification in field theoretical models [4, 5, 6]. Similarly, as pointed out in Introduction, confined fields can be treated with TFD by choosing appropriately the parameter α in the Bogoliubov transformation [8, 9, 10]. To see how this works, replace β and k_0 in Eq. (12) by $\alpha = i2L$ and k_3 , corresponding to confinement along the z -axis, writing

$$v_k^2(L) = \sum_{l=1}^{\infty} (-\xi)^{l+1} e^{-i2Lk_3 l}. \quad (17)$$

Using this v_k^2 in Eq. (6) and performing the inverse Fourier transform, we get

$$\begin{aligned} \bar{G}_0^{11}(x-x';L) &= \sum_{l=1}^{\infty} (-\xi)^{l+1} [G_0^*(x'-x-2Ll\hat{n}_3) \\ &\quad - G_0(x-x'-2Ll\hat{n}_3)] \end{aligned} \quad (18)$$

where $\bar{G}_0^{11}(x-x';L) = G_0^{11}(x-x';L) - G_0(x-x')$ and $\hat{n}_3 = (0, 0, 0, 1)$. For this situation, similar steps as those leading to Eqs. (15) and (16) give

$$\mathcal{T}_M^{\mu\nu(11)}(L) = -\frac{2}{\pi^2} \sum_{l=1}^{\infty} \frac{g^{\mu\nu} + 4\hat{n}_3^\mu\hat{n}_3^\nu}{(2Ll)^4} = -\frac{\pi^2}{720L^4} (g^{\mu\nu} + 4\hat{n}_3^\mu\hat{n}_3^\nu), \quad (19)$$

$$\mathcal{T}_D^{\mu\nu(11)}(L) = \frac{4}{\pi^2} \sum_{l=1}^{\infty} (-1)^l \left[\frac{g^{\mu\nu} + 4\hat{n}_3^\mu\hat{n}_3^\nu}{(2Ll)^4} \right] = -\frac{7\pi^2}{2880} [g^{\mu\nu} + 4\hat{n}_3^\mu\hat{n}_3^\nu]. \quad (20)$$

The Casimir effect for the electromagnetic field between parallel metallic plates can be obtained from Eq. (19); the Casimir energy and pressure are

$$E(L) = \mathcal{T}_M^{00(11)}(L) = -\frac{\pi^2}{720L^4}, \quad P(L) = \mathcal{T}_M^{33(11)}(L) = -\frac{\pi^2}{240L^4}.$$

Similarly, from Eq. (20), we find the Casimir energy and pressure for the Dirac field confined between parallel plates, with anti-periodic boundary conditions, as:

$$E(L) = \mathcal{T}_D^{00(11)}(L) = -\frac{7\pi^2}{2880L^4}; \quad P(L) = \mathcal{T}_D^{33(11)}(L) = -\frac{7\pi^2}{960L^4}.$$

These results demonstrate explicitly the usefulness of the Bogoliubov transformation to treat confined fields as a generalization of TFD. From the above considerations, a question emerges naturally: what should be the appropriate generalization of the Bogoliubov transformation to account for simultaneously space compactification and thermal effects?

Such a generalization must reproduce, for example, the known results for the Casimir effect in the case of the parallel plates geometry at finite temperatures. Since energy is an additive quantity, we expect to have L - and T -dependent contributions plus a mixed (LT -dependent) contribution representing the interference of the two effects. In the next Section, we will show that the proper extension of expressions (12) and (17), for this case, is

$$v_k^2(\beta, L) = \sum_{l_0=1}^{\infty} (-\xi)^{l_0+1} e^{-\beta k_0 l_0} + \sum_{l_3=1}^{\infty} (-\xi)^{l_3+1} e^{-i2Lk_3 l_3} + 2 \sum_{l_0, l_3=1}^{\infty} (-\xi)^{l_0+l_3+2} e^{-\beta k_0 l_0 - i2Lk_3 l_3}. \quad (21)$$

To treat the general situation, compatible with cartesian geometries, we will consider the $(1+N)$ -dimensional Minkowski space. Then, taking $\alpha = (\alpha_0, \alpha_1, \alpha_2, \dots, \alpha_N)$, we write

$$v_k^2(\alpha) = \sum_{s=1}^{N+1} \sum_{\{\sigma_s\}} \left(\prod_{n=1}^s f(\alpha_{\sigma_n}) \right) 2^{s-1} \times \sum_{l_{\sigma_1}, \dots, l_{\sigma_s}=1}^{\infty} (-\xi)^{s+\sum_{r=1}^s l_{\sigma_r}} \exp\left\{-\sum_{j=1}^s \alpha_{\sigma_j} l_{\sigma_j} k_{\sigma_j}\right\}, \quad (22)$$

where $f(\alpha_j) = 0$ for $\alpha_j = 0$, $f(\alpha_j) = 1$ otherwise and $\{\sigma_s\}$ denotes the set of all combinations with s elements, $\{\sigma_1, \sigma_2, \dots, \sigma_s\}$, of the first $N+1$ natural numbers $\{0, 1, 2, \dots, N\}$, that is all subsets containing s elements, which we choose to write in an ordered form with $\sigma_1 < \sigma_2 < \dots < \sigma_s$. Inserting this $v_k^2(\alpha)$ into Eq. (6) and taking the inverse Fourier transform, we obtain

$$\bar{G}_0^{11}(x-x'; \alpha) = \sum_{s=1}^{N+1} \sum_{\{\sigma_s\}} \left(\prod_{n=1}^s f(\alpha_{\sigma_n}) \right) \sum_{l_{\sigma_1}, \dots, l_{\sigma_s}=1}^{\infty} (-\xi)^{s+\sum_{r=1}^s l_{\sigma_r}} \times 2^{s-1} \left[G_0^*(x' - x - i \sum_{j=1}^s \eta_{\sigma_j} \alpha_{\sigma_j} l_{\sigma_j} \hat{n}_{\sigma_j}) - G_0(x - x' - i \sum_{j=1}^s \eta_{\sigma_j} \alpha_{\sigma_j} l_{\sigma_j} \hat{n}_{\sigma_j}) \right] \Bigg|_{x' \rightarrow x}, \quad (23)$$

where $\eta_{\sigma_j} = +1$, if $\sigma_j = 0$, and $\eta_{\sigma_j} = -1$ for $\sigma_j = 1, 2, \dots, N$. To get the physical situation at finite temperature and spatial confinement, α_0 has to be taken as a positive real number while α_n , for $n = 1, 2, \dots, N$, must be pure imaginary of the form $i2L_n$; in these cases, one finds that $\alpha_j^{*2} = \alpha_j^2$.

Considering such choices for parameters α_j and using the explicit form of $\bar{G}_0^{11}(x-x'; \alpha)$ in the 4-dimensional space-time (corresponding to $N = 3$), we obtain the renormalized α -dependent

energy-momentum tensor in the general case, for both Maxwell and Dirac fields:

$$\begin{aligned} \mathcal{T}_M^{\mu\nu(11)}(\alpha) &= -i \left\{ \Gamma^{\mu\nu}(x, x') \bar{G}_0^{11}(x - x'; \alpha) \right\} \Big|_{x \rightarrow x'} \\ &= -\frac{2}{\pi^2} \sum_{s=1}^4 \sum_{\{\sigma_s\}} \left(\prod_{n=1}^s f(\alpha_{\sigma_n}) \right) 2^{s-1} \\ &\quad \times \sum_{l_{\sigma_1}, \dots, l_{\sigma_s}=1}^{\infty} \left[\frac{g^{\mu\nu}}{[\sum_{j=1}^s \eta_{\sigma_j} (\alpha_{\sigma_j} l_{\sigma_j})^2]^2} \right. \\ &\quad \left. - \frac{2 \sum_{j,r=1}^s (1 + \eta_{\sigma_j} \eta_{\sigma_r}) (\alpha_{\sigma_j} l_{\sigma_j}) (\alpha_{\sigma_r} l_{\sigma_r}) \hat{n}_{\sigma_j}^\mu \hat{n}_{\sigma_r}^\nu}{[\sum_{j=1}^s \eta_{\sigma_j} (\alpha_{\sigma_j} l_{\sigma_j})^2]^3} \right]; \end{aligned} \quad (24)$$

$$\begin{aligned} \mathcal{T}_D^{\mu\nu(11)}(\alpha) &= -4i \partial^\mu \partial^\nu [\bar{G}_0^{11}(x - x'; \alpha)]_{x' \rightarrow x} \\ &= -\frac{4}{\pi^2} \sum_{s=1}^4 \sum_{\{\sigma_s\}} \left(\prod_{n=1}^s f(\alpha_{\sigma_n}) \right) \sum_{l_{\sigma_1}, \dots, l_{\sigma_s}=1}^{\infty} (-1)^{s+\sum_{r=1}^s l_{\sigma_r}} \\ &\quad \times 2^{s-1} \left[\frac{g^{\mu\nu}}{[\sum_{j=1}^s \eta_{\sigma_j} (\alpha_{\sigma_j} l_{\sigma_j})^2]^2} \right. \\ &\quad \left. - \frac{2 \sum_{j,r=1}^s (1 + \eta_{\sigma_j} \eta_{\sigma_r}) (\alpha_{\sigma_j} l_{\sigma_j}) (\alpha_{\sigma_r} l_{\sigma_r}) \hat{n}_{\sigma_j}^\mu \hat{n}_{\sigma_r}^\nu}{[\sum_{j=1}^s \eta_{\sigma_j} (\alpha_{\sigma_j} l_{\sigma_j})^2]^3} \right]. \end{aligned} \quad (25)$$

Notice that the results obtained so far (Eqs. (15) and (19) for the Maxwell field and Eqs. (16) and (20) for the Dirac field) are particular cases of the above expressions, corresponding to $\alpha = (\beta, 0, 0, 0)$ and $\alpha = (0, 0, 0, i2L)$ respectively. Another important aspect is that $\mathcal{T}^{\mu\nu(11)}(\alpha)$ is traceless in both cases, as it should be. Now, we will apply these general results to some specific examples.

4. Casimir effect for parallel plates at finite temperature

As the first example of the development of the last Section, we now consider the electromagnetic field satisfying the Dirichlet boundary condition on parallel planes (metallic plates), normal to the z -direction, at finite temperature. In this case, $v_k^2(\alpha)$ is given by Eq. (21) with $\xi = -1$ (corresponding to the choice $\alpha = (\beta, 0, 0, i2L)$) and Eq. (24) reduces to

$$\begin{aligned} \mathcal{T}_M^{\mu\nu(11)}(\beta, L) &= -\frac{2}{\pi^2} \left\{ \sum_{l_0=1}^{\infty} \frac{g^{\mu\nu} - 4\hat{n}_0^\mu \hat{n}_0^\nu}{(\beta l_0)^4} + \sum_{l_3=1}^{\infty} \frac{g^{\mu\nu} + 4\hat{n}_3^\mu \hat{n}_3^\nu}{(2Ll_3)^4} \right. \\ &\quad \left. + 2 \sum_{l_0, l_3=1}^{\infty} \frac{(\beta l_0)^2 [g^{\mu\nu} - 4\hat{n}_0^\mu \hat{n}_0^\nu] + (2Ll_3)^2 [g^{\mu\nu} + 4\hat{n}_3^\mu \hat{n}_3^\nu]}{[(\beta l_0)^2 + (2Ll_3)^2]^3} \right\}. \end{aligned} \quad (26)$$

It follows then that the Casimir energy ($\mathcal{T}_M^{00(11)}$) and pressure ($\mathcal{T}_M^{33(11)}$) are given by [8]

$$E(\beta, L) = \frac{\pi^2}{15\beta^4} - \frac{\pi^2}{720L^4} + \frac{4}{\pi^2} \sum_{l_0, l_3=1}^{\infty} \frac{3(\beta l_0)^2 - (2Ll_3)^2}{[(\beta l_0)^2 + (2Ll_3)^2]^3}, \quad (27)$$

$$P(\beta, L) = \frac{\pi^2}{45\beta^4} - \frac{\pi^2}{240L^4} + \frac{4}{\pi^2} \sum_{l_0, l_3=1}^{\infty} \frac{(\beta l_0)^2 - 3(2Ll_3)^2}{[(\beta l_0)^2 + (2Ll_3)^2]^3}. \quad (28)$$

The first two terms of these expressions reproduce Eqs. (15) and (19), giving the blackbody and the Casimir contributions for the energy and the pressure, separately. The last term represents the interplay between the two effects. These results have been obtained before with the use of mode-sum techniques and the image method [?, 18].

Notice that the positive black-body contributions for the energy E and pressure P dominate in the high-temperature limit, while the energy and the pressure are negative for low T . From Eq. (28), we determine the critical curve ($\beta_c = \chi_0 L$) for the transition from negative to positive values of P , by searching for the value of the ratio $\chi = \beta/L$ for which the pressure vanishes; this value, χ_0 , is the solution of the transcendental equation

$$\frac{\pi^2}{45} \frac{1}{\chi^4} - \frac{\pi^2}{240} + \frac{4}{\pi^2} \sum_{l,n=1}^{\infty} \frac{(\chi l)^2 - 3(2n)^2}{[(\chi l)^2 + (2n)^2]^3} = 0, \quad (29)$$

given, numerically, by $\chi_0 \simeq 1.316$. In the next section we explore such an analysis where four spatial dimensions are compactified.

5. Casimir effect for the fermion sector of a non-interacting massless QCD

In this section we analyze the Casimir effect for a simplified quantum chromodynamics (QCD) model. The QCD Lagrangian is given by

$$\mathcal{L} = \bar{\psi}(x)[iD_\mu \gamma^\mu - m]\psi(x) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\alpha}(\partial^\mu A_\mu^r(x))^2 + A_\mu^r(x)t^r J^\mu(x),$$

where

$$F_{\mu\nu}^r = \partial_\mu A_\nu^r(x) - \partial_\nu A_\mu^r(x) + g c^{rst} A_\mu^s(x) A_\nu^t(x),$$

$F_{\mu\nu} = \sum_r F_{\mu\nu}^r t^r$, is the field tensor describing the gluons; t^r and c^{rst} are, respectively, the generators and the structure constants of $SU(3)$; $D_\mu = \partial_\mu + igA_\mu = I\partial_\mu + igA_\mu^r(x)t^r$ is the covariant derivative; $\psi(x)$ stands for the quark field, including the flavor and color components. The term $\frac{1}{2\alpha}(\partial^\mu A_\mu^r(x))^2$ is the gauge fixing term.

We consider an approximation for \mathcal{L} describing some features of a massless baryon-free quark-gluon plasma, confined in space under the static bag model condition, corresponding to the adiabatic expansion of the plasma. In this case one discards, as a zero-order approximation, the interactions and the quark mass. Then the energy-momentum tensor for the quark field is given by

$$\begin{aligned} T_q^{\mu\nu}(x) &= \{in_c \sum_f \bar{\psi}(x)\gamma^\mu \partial^\nu \psi(x')\}_{|x' \rightarrow x}, \\ &= \{in_c n_f \bar{\psi}(x)\gamma^\mu \partial^\nu \psi(x')\}_{|x' \rightarrow x}, \end{aligned}$$

where n_c and n_f are the number of colors and flavors in the $SU(3)$ non-abelian gauge theory. With $T_q^{\mu\nu}(x)$ explicitly written, we introduce $\mathcal{T}_q^{\mu\nu(ab)}(x; \alpha)$ which is given by

$$\mathcal{T}_q^{\mu\nu(ab)}(x; \alpha) = \langle T_q^{\mu\lambda(ab)}(x; \alpha) \rangle - \langle T_q^{\mu\lambda(ab)}(x) \rangle,$$

where

$$\begin{aligned} \langle T_q^{\mu\lambda(ab)}(x) \rangle &= -i4n_c n_f \partial^\nu \partial^\mu G_0^{(ab)}(x - x')|_{x' \rightarrow x}, \\ \langle T_q^{\mu\lambda(ab)}(x; \alpha) \rangle &= -i4n_c n_f \partial^\nu \partial^\mu G_0^{(ab)}(x - x'; \alpha)|_{x' \rightarrow x}, \end{aligned}$$

and so, for the quark field

$$\begin{aligned}
T_q^{\mu\nu(11)}(\alpha) &= -i4n_c n_f \partial^\mu \partial^\nu [\bar{G}_0^{11}(x-x'; \alpha)]_{x' \rightarrow x} \\
&= -\frac{4n_c n_f}{\pi^2} \sum_{s=1}^4 \sum_{\{\sigma_s\}} \left(\prod_{n=1}^s f(\alpha_{\sigma_n}) \right) \sum_{l_{\sigma_1}, \dots, l_{\sigma_s}=1}^{\infty} (-1)^{s+\sum_{r=1}^s l_{\sigma_r}} \\
&\quad \times 2^{s-1} \left[\frac{g^{\mu\nu}}{[\sum_{j=1}^s \eta_{\sigma_j} (\alpha_{\sigma_j} l_{\sigma_j})^2]^2} \right. \\
&\quad \left. - \frac{2 \sum_{j,r=1}^s (1 + \eta_{\sigma_j} \eta_{\sigma_r}) (\alpha_{\sigma_j} l_{\sigma_j}) (\alpha_{\sigma_r} l_{\sigma_r}) \hat{n}_{\sigma_j}^\mu \hat{n}_{\sigma_r}^\nu}{[\sum_{j=1}^s \eta_{\sigma_j} (\alpha_{\sigma_j} l_{\sigma_j})^2]^3} \right]. \tag{30}
\end{aligned}$$

For gluon field, in the approximation considered here, we have basically the same tensor for the electromagnetic field up to the color number, n_g . For this reason, we focus our analysis on the fermion sector only.

Taking the system in a Γ_4^4 topology, the quark field is physically analyzed in a cubic box of edge L at finite temperature, T ($\beta = T^{-1}$), satisfying anti-periodic boundary conditions in the four dimensions. In this case we have $\alpha = (\beta, iL, iL, iL)$ and

$$T_q^{33(11)}(\beta, L) = n_c n_f f(\chi) \frac{1}{L^4}$$

where

$$\begin{aligned}
f(\chi) &= \frac{1}{\pi^2} \left\{ \mathcal{C}_f + \frac{7\pi^4}{180} \frac{1}{\chi^4} + 16 \sum_{l,n=1}^{\infty} \frac{(-1)^{l+n}}{[\chi^2 l^2 + n^2]^2} \right. \\
&\quad + 8 \sum_{l,n=1}^{\infty} (-1)^{l+n} \frac{\chi^2 l^2 - 3n^2}{[\chi^2 l^2 + n^2]^3} - 16 \sum_{l,n,r=1}^{\infty} \frac{(-1)^{l+n+r}}{[\chi^2 l^2 + n^2 + r^2]^2} \\
&\quad - 32 \sum_{l,n,r=1}^{\infty} (-1)^{l+n+r} \frac{\chi^2 l^2 + n^2 - 3r^2}{[\chi^2 l^2 + n^2 + r^2]^3} \\
&\quad \left. + 32 \sum_{l,n,r,q=1}^{\infty} (-1)^{l+n+r+q} \frac{\chi^2 l^2 + n^2 + r^2 - 3q^2}{[\chi^2 l^2 + n^2 + r^2 + q^2]^3} \right\}
\end{aligned}$$

with $\chi = \frac{\beta}{L}$ and

$$\mathcal{C}_f = -8 \sum_{l,n=1}^{\infty} \frac{(-1)^{l+n}}{[l^2 + n^2]^2} + \frac{16}{3} \sum_{l,n,r=1}^{\infty} \frac{(-1)^{l+n+r}}{[l^2 + n^2 + r^2]^2} - \frac{7\pi^4}{180} \approx -5.67$$

The pressure changes from negative to positive as the temperature is increased. Indeed, for $T \rightarrow 0$, we have

$$T_q^{33(11)}(L) = n_c n_f \frac{\mathcal{C}_f}{\pi^2} \frac{1}{L^4} < 0.$$

For $T \rightarrow \infty$, the pressure is dominated by the term $\simeq T^4$, which is positive. The value of χ for which the pressure changes from negative to positive is

$$\chi_c \approx 1.00 \Rightarrow T_c(L) \simeq \frac{1}{L}$$

so that for $L = 1\text{fm} \longrightarrow T_c(L) \simeq 200\text{ MeV}$. This result is indicative that the Casimir effect plays a role in the deconfinement process of the hadronic matter. Although this role has been pointed out before [18], we find this by a different method. This fact reinforces the use of the generalized Bogoliubov transformation, that describes the Casimir effect as a condensate.

6. Concluding remarks

In this paper we have used generalizations of the TFD Bogoliubov transformation, describing fields in topology of the type $\Gamma_D^d = S^{1_1} \times \dots \times S^{1_d} \times R^{D-d}$, to address the Casimir effect at finite temperature for both boson and fermion fields. It is worth emphasize that the procedure developed here is simpler and more direct than the standard techniques, such as the sum of modes and the image method, in particular to introduce the renormaliation procedure. This is a direct consequence of the structure of the Green function, which is written in two parts: one describing the divergent term due to the free space-time and the other due to the compactification effect. The result is that physical quantities are introduced as functions of the temperature and the compactification lengths. Notice that this procedure can describe systems satisfying other boundary conditions than those ones studied here. In this case, other types of Bogoliubov transformation describing fields in Γ_D^d can be defined, in particular, as the counterpart of the method studied in Ref. [1] for twisted boundary conditions.

As an application of this technique, we have analyzed the fermion sector of a massless baryon-free quark-gluon plasma, confined in a topology Γ_D^d , corresponding to an adiabatic expansion of the plasma. The main result is that by raising the temperature, there is a critical value for which the pressure changes sign. The transition from negative to positive Casimir pressure is at $T \simeq 200\text{Mev}$, for a length $L \simeq 1\text{ fm}^{-1}$. This result is another way to show the importance of the Casimir pressure for the deconfinement of the hadronic matter. A detailed analysis of these aspects and the nature of the generalized Bogoliubov transformation, including a discussion about scalar twisted fields, will be considered elsewhere.

Acknowledgments

Financial support from CNPq (Brazil) and NSERC (Canada) is acknowledged.

References

- [1] Birrell N D and Ford L H, Phys. Rev. D **22**, 330 (1980).
- [2] Ford L H and Svaiter N F, Phys. Rev D **51**, 6981 (1995).
- [3] Ford L H and Yoshimura T, Phys. Lett. A **70**, 89 (1979).
- [4] Malbouisson A P C and Malbouisson J M C, J. Phys. A: Math. Gen. **35**, 2263 (2002).
- [5] Malbouisson A P C, Malbouisson J M C and Santana A E, Nucl. Phys. B **631**, 83 (2002).
- [6] Malbouisson A P C, Malbouisson J M C, Santana A E and da Silva J C, Phys. Lett. B **583**, 373 (2004).
- [7] Malbouisson A P C, Khanna F C, Malbouisson J M C and Santana A E, **36**, 1165 (2006).
- [8] da Silva J C, Khanna F C, Matos Neto A and Santana A E, Phys. Rev. A **66**, 052101 (2002) .
- [9] Queiroz H, da Silva J C, Khanna F C, Malbouisson J M C, Revzen M and Santana A E, Ann. Phys. (N.Y.) **317**, 220 (2005).
- [10] Santana A E, Malbouisson J M C, Malbouisson A P C and Khanna F C, *Thermal Field Theory: Algebraic Aspects and Applications to Confined Systems*, and Khanna F C, Malbouisson J M C and Santana A E, *Thermofield Dynamics: Generalized Bogoliubov Transformation and Applications to the Casimir Effect*, in Non-Linear Dynamics and Fundamental Interactions, Khanna F and Matrasulov D Eds. (Kluwer/Springer, N. York, 2005).
- [11] Milonni P W, *The Quantum Vacuum* (Academic, Boston, 1993).
- [12] Mostepanenko V M and Trunov N N, *The Casimir Effect and its Applications* (Clarendon, Oxford, 1997).
- [13] Bordag M, Mohideen U and Mostepanenko V M, *New Developments in Casimir Effect*, quant-ph/0106045, Phys. Rep. **353**, 1 (2001).
- [14] Levin F S and Micha D A (Eds.), *Long Range Casimir Forces* (Plenum, New York, 1993).
- [15] Seife C, Science **275**, 158 (1997).
- [16] Boyer T H, Am. J. Phys. **71**, 990 (2003).

- [17] Milton K A, *The Casimir Effect: Physical Manifestations of Zero Point Energy*, hep-th/9901011 (W. Scientific, Singapore, 2001).
- [18] Plunien G, Müller B and Greiner W, Phys. Rep. **134**, 87(1986).
- [19] Saito K, Z. Phys. C **50**, 69 (1991).
- [20] Gundersen S A and Ravndal F, Ann. Phys. (N.Y.) **182**, 90 (1988).
- [21] Lütken C A and Ravndal F, J. Phys. A: Math. Gen. **21**, L793 (1988).
- [22] Lütken C A and Ravndal F, J. Phys. G: Nucl. Phys. **10**, 123 (1984).
- [23] Ravndal F and Tollefsen D, Phys. Rev. D **40**, 4191 (1989).
- [24] De Paola R D M, Rodrigues R B and Svaiter N F, Mod. Phys. Lett. **44**, 2353 (1999).
- [25] Takahashi Y and Umezawa H, Coll. Phenomena **2** (1975) 55 (Reprinted in Int. J. Mod. Phys. 10, 1755 (1996)).
- [26] Umezawa H, Matsumoto H and Tachiki M, *Thermofield Dynamics and Condensed States* (North-Holland, Amsterdam, 1982).
- [27] Umezawa H, *Advanced Field Theory: Micro, Macro and Thermal Physics* (AIP, New York, 1993).