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# Multifractal analysis of plasmon–polariton and light transmission spectra in quasiperiodic multilayers

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**Abstract**

We investigate the multifractal properties for the spectra of plasmon–polaritons and light waves in multiple layers arranged in a quasiperiodic fashion. The profiles of the plasmon polariton bandwidths and the light transmission spectra are analyzed and the $f(\alpha)$ functions are calculated for different physical situations. The quasiperiodic structures considered here are the Fibonacci, Thue–Morse and double-period sequences, which are characterized by their Fourier spectra. For the polaritons case, we show that the $f(\alpha)$ functions have a slightly variation for different values of the dimensionless in-plane wavevector $k_x a$. We also show that the $f(\alpha)$ functions of the light transmission spectra are almost independent of the quasiperiodic sequences. © 1999 Elsevier Science B.V. All rights reserved.

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## 1. Introduction

Since the discovery of the icosahedral phase in Al–Mn alloys by Shechtman et al. [1] and the experimental growth of the first Fibonacci superlattice by Merlin et al. [2], much attention have been given for the so called *quasiperiodic multilayer systems*. These quasiperiodic structures are formed by the superposition of two (or more) incommensurate periods, so that they can be defined as intermediate systems between a periodic crystal and the random amorphous solid [3–5]. Theoretical works

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have demonstrated that physical properties like critical states, Cantor set spectrum, power-law behavior, etc. are proper features of these structures [6,7].

A common (and more fascinating) feature of these structures is a *fractal* or a *multifractal* spectra of energy, forming a Cantor set, which are their *basic signatures*. Previous works in semiconductor [8] and magnetic [9,10] materials, have shown these fractal spectra and their scaling properties.

Fractality (or multifractality) is in general a common property of strange attractors in nonlinear systems [11,12]. In nonlinear physics a natural wish is to characterize these objects and to describe the events occurring on them. Therefore, the study of the $f(\alpha)$ function is very important. It describes the distribution of different fractal dimensions of the object upon variation of the singularities of strength $\alpha$ [13]. In spite that quasiperiodic systems are not classifiable in the nonlinear physics context, they also exhibit multifractality in their spectra. This feature was firstly studied by Kohmoto and collaborators to electrons in one-dimensional quasiperiodic discontinuous potential [14]. Recently Bezerra et al. have reported a multifractal analysis of the spin wave spectra in quasiperiodic magnetic multilayer [15]. They have shown that the profiles of these spectra are multifractal, with a well defined $f(\alpha)$ function. Moreover, they have demonstrated that the $f(\alpha)$ functions have a slight dependence in the in-plane wavevector.

In a previous work we have reported the distribution of the bandwidth of the plasmon–polariton spectra in quasiperiodic semiconductor multilayers [8]. We have found a power law associated with the number of layers of the system. For example, the power law associated with Fibonacci semiconductor multilayer system is $\Delta \sim F_n^{-\delta}$, where $\Delta$ is the sum of the bandwidth of all allowed bands in a given generation of the Fibonacci sequence, $F_n$ is the Fibonacci number which also is the number of the layers in the unit cell, and $\delta$ is an exponent which is depending of the in-plane wavevector $k_x a$, $a$ being the lattice constant. This exponent can indicate the localization degree of the excitation [16]. Its multifractal analyses is therefore a complement of this work.

It is the aim of this work to find and characterize the $f(\alpha)$ functions for the plasmon–polariton and light transmission spectra for three types of quasiperiodic multilayers: Fibonacci, Thue–Morse and double-period. Let us briefly review their properties: First we recall the definition of a *substitutional sequence*. Take a finite set $\xi$ (here $\xi = \{A, B\}$) called an *alphabet* and denote by $\xi^*$ the set of all finitely long words that can be written in this alphabet. Now let $\zeta$ be a map from $\xi$ to $\xi^*$ by specifying that $\zeta$ acts on a word by substituting each letter (e.g. $A$) of this word by its corresponding image $\zeta(A)$. A sequence is then called a *substitutional sequence* if it is a fixpoint of $\zeta$, i.e. it remains invariant if each letter in the sequence is replaced by its image under $\zeta$. As examples of substitutional sequences that have attracted most attention in physics we have (all of them with $\xi = \{A, B\}$): (a) The Fibonacci sequence, where the substitution rules are $A \to \zeta(A) = AB$, $B \to \zeta(B) = A$; (b) The Thue–Morse sequence, with the rules where $A \to \zeta(A) = AB$, $B \to \zeta(B) = BA$; (c) The double-period sequence, where $A \to \zeta(A) = AB$, $B \to \zeta(B) = AA$.

Fractal and multifractal objects have been identified in many physical situations, ranging from problems of aggregation to the behavior of chaotic dynamical systems [17,18]. The multifractal sets have been characterized on the basis of the generalized dimensions $D_q$ and the associated spectrum of singularities $f(\alpha)$, and they can be completely described either by an infinite number of the generalized dimensions $D_q$ or by the spectrum of singularities $f(\alpha)$ [19–22]. The curve $D_q$ versus $q$ is defined by the expression:

$$D_q = \frac{1}{q-1} \lim_{N' \to \infty} \left\{ -\frac{\ln \sum_i p_i^q}{\ln N'} \right\} , \tag{1}$$

with $D_{q=1} = D_1$ given by

$$D_1 = \lim_{N' \to \infty} \left\{ -\frac{\sum_i p_i \ln p_i}{\ln N'} \right\} . \tag{2}$$

Here $p_i = \int_{\text{box}} d\mu$, $\mu$ being the probability measure of the multifractal set, and $i = 1, 2, \ldots, N'$ ($N'$ is the number of boxes). Also, $i$ is the index of a box that belongs to a grid that covers the set and has a linear size $\varepsilon = 1/N'$.

The scaling exponent $\alpha$ is defined by

$$\alpha(x) = \lim_{N' \to \infty} \left\{ -\frac{\ln p(x)}{\ln N'} \right\} , \tag{3}$$

where $p(x)$ is the integral of $\mathrm{d}\mu$ over a box with center in $x$. The $f(\alpha)$ function is then defined by the relation

$$N'(\alpha, \varepsilon) \sim \varepsilon^{-f(\alpha)} \tag{4}$$

for $\varepsilon \to 0$. In this equation, $N'(\alpha, \varepsilon)$ is the number of boxes $\varepsilon$ with $\alpha$ between $\alpha$ and $\alpha + \Delta\alpha$.

There are various numerical procedure to calculate the $f(\alpha)$ function. One of most efficient algorithm was developed by Chhabra and Jensen [23,24]. It allows us to obtain the $f(\alpha)$ function with an excellent numerical precision, and it is the method used here.

The plan of this paper is as follows: in Section 2, we present a brief review of the plasmon–polariton properties in the quasiperiodic multilayer systems considered here, including their bandwidths for fixed values of the in-plane dimensionless wavevector $k_x a$. Then we present the $f(\alpha)$ functions to characterize their multifractal profile. In Section 3, we relate the method used to calculate the light transmission spectra as well as the determination of their $f(\alpha)$ functions. The conclusions are in Section 4.

## 2. Plasmon–polariton spectra

To set up a quasiperiodic semiconductor multilayer, we consider two different building blocks, $A$ and $B$, which are arranged in a desired way depending on the quasiperiodic sequence. The building block $A(B)$ consists of a two-dimensional electron gas (2DEG) with a carrier concentration $n_A(n_B)$ supported by a dielectric layer $A(B)$. The layers $A$ and $B$ are characterized by the dielectric functions $\varepsilon_A(\omega)$ and $\varepsilon_B(\omega)$, and have thicknesses $a$ and $b$, respectively.

In order to find the bulk polariton modes, we consider the infinite structure of repeated units cell, where the Cartesian axis are chosen in such a way that the $z$-axis is normal to the $xy$-plane of the layers. The substitutional rules explained above should be applied in the building blocks for each unit cell. Let us assume that the propagation of the electromagnetic wave is $p$-polarized. The 2DEG at each interface is modeled due to the presence of a surface density of current whose expression, given by Ohm's law, is

$$J_{x\xi} = i\omega\varepsilon_0\sigma_\xi E_{x\xi} \,, \tag{5}$$

where

$$\sigma_\xi = \frac{n_\xi e^2}{m^*\omega(\omega + i\gamma_\xi)} \,, \tag{6}$$

where, $\xi = A$ or $B$, $n_\xi$ is the carrier concentration, $e(m^*)$ is the electron's charge (effective mass), $\varepsilon_0$ is the vacuum permissivity, and $\gamma_\xi$ is the damping factor of the material.

To find the polariton dispersion relation, one should solve the electromagnetic wave equation within the layers $A$ and $B$ of the $N$th unit cell of the multilayer system. Then, by using Maxwell's boundary conditions together with a transfer matrix treatment and Bloch's ansatz, we obtain the dispersion relation for the polariton modes, i.e.:

$$\cos(QL) = (1/2)\mathrm{Tr}(T_{S_N}) \,, \tag{7}$$

where $Q$ is the Bloch wave vector and $L$ is the thickness of the unit cell. Here, the form of transfer matrix $T_{S_N}$ is different for each type of quasiperiodic system. These matrices can be found elsewhere [8].

In Fig. 1, we show the plasmon–polariton allowed bandwidths for the various quasiperiodic structures and for $k_x a = 0.25$. We consider medium $A$ as GaAs and medium $B$ as $SiO_2$. Instead of using the frequency $\omega$, we prefer to replace it by the reduced frequency $\omega/\Omega$ where $\Omega$ is given by

$$\Omega = \left( \frac{\varepsilon_{\infty A} n_A e^2}{m^* \varepsilon_0 a} \right)^{1/2} . \tag{8}$$

For GaAs the value of $\Omega$ is approximately equal to 23 THz.

To determine the corresponding multifractal spectra for the profiles of plasmon–polariton bandwidths, we first define a measure by the normalized local plasmon–polariton bandwidths ($\Delta_i$), i.e.:

$$\xi_i = \frac{\Delta_i}{\sum_i \Delta_i} \,, \tag{9}$$

and construct a parametrized family of normalized measures defined by

$$\mu_i = \frac{\xi_i^q}{\sum_i \xi_i^q} \,, \tag{10}$$

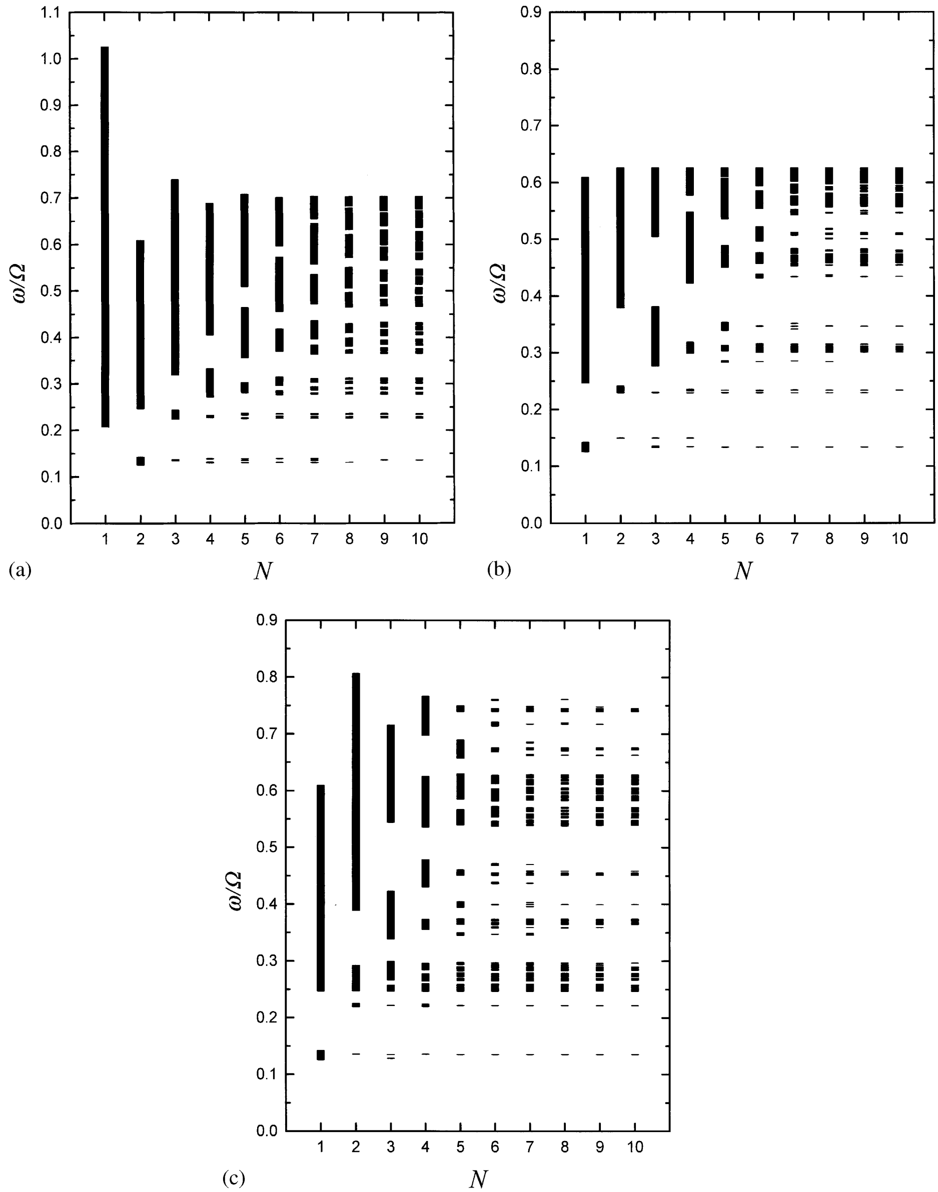


Fig. 1. The distribution of the plasmon–polariton allowed bandwidths for the quasiperiodic structures considered here as function of the generation number $N$. (a) Fibonacci; (b) Thue–Morse; (c) double-period.

which is a generalization of the original measure $\xi_i$. The spectrum $f(\alpha)$ is obtained by varying the parameter $q$ and calculating

$$f(\alpha_q) = \lim_{N' \to \infty} \left\{ -\frac{\sum_i \mu_i \ln \mu_i}{\ln N'} \right\} , \tag{11}$$

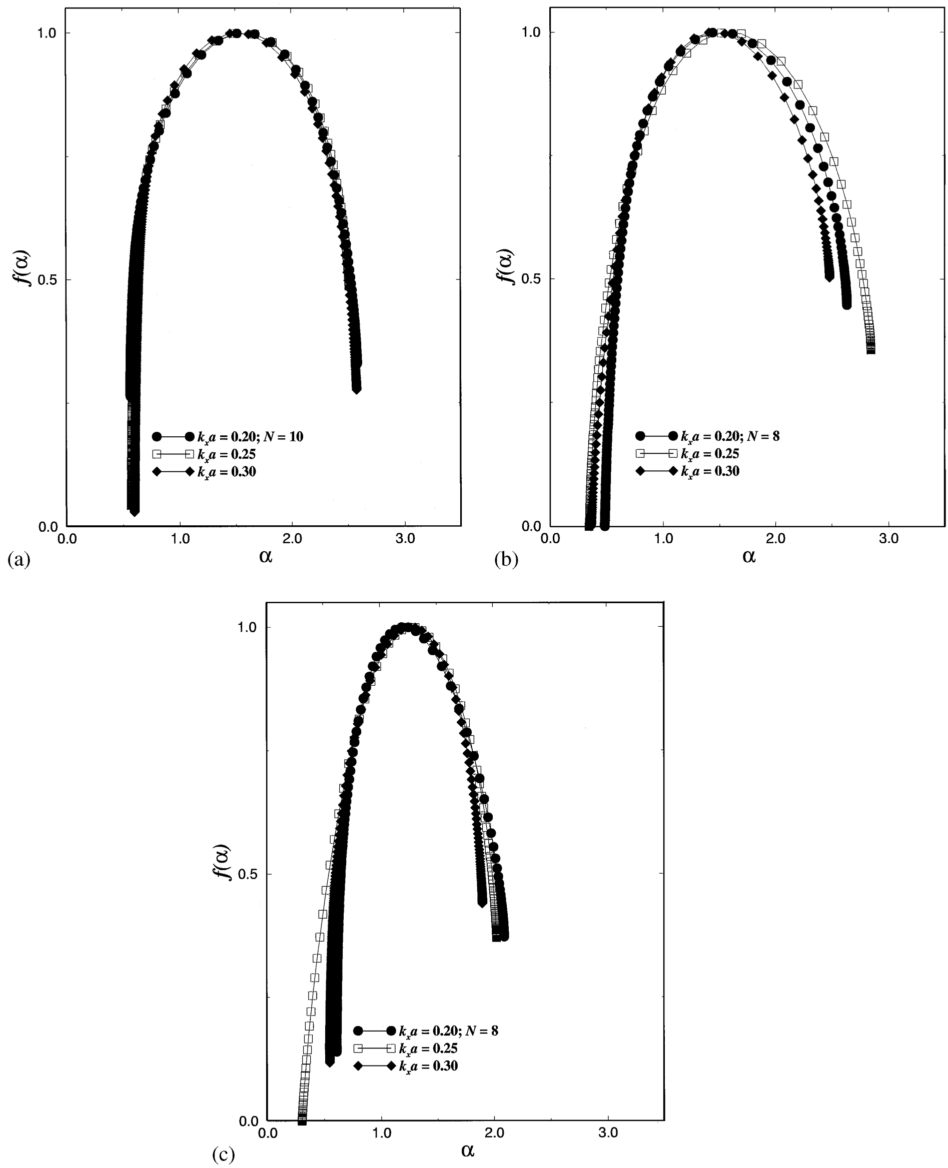


Fig. 2. The $f(\alpha)$ functions of the plasmon–polariton bandwidths for the quasiperiodic structures considered here. The values of $k_xa$ are given at the legend. (a) Fibonacci; (b) Thue–Morse; (c) double-period.

$$\alpha_q = \lim_{N'\to\infty} \left\{ -\frac{\sum_i \mu_i \ln \xi_i}{\ln N'} \right\} . \tag{12}$$

Fig. 2 shows the $f(\alpha)$ functions for three different values of the dimensionless in-plane wavevector $k_xa$, namely 0.2, 0.25 and 0.3. In Fig. 2a, we present $f(\alpha)$ for the 10th generation of the Fibonacci sequence, which implies a unit cell with 55 A's and 34 B's building layers. We notice that the spectra is independent of the different values of $k_x a$. Fig. 2b shows the $f(\alpha)$ functions for the Thue–Morse superlattice. We consider here its 8th generation, whose unit cell is composed by 128 A's and 128 B's building layers. For this structure we can infer a small variations of the spectra for the different values of $k_x a$, although the $f(\alpha)$ spectra widths are almost the same. The $f(\alpha)$ functions for the 8th double-period generation, are shown in Fig. 2c, where the unit cell has now 161 A's and 85 B's building layers. As in the Thue–Morse case, we notice small variations of the spectra in respect of different values of the in-plane wavevectors. In all curves, the extremes $\alpha_{\min}$ and $\alpha_{\max}$ of the abscissa of the $f(\alpha)$ curves, represent the minimum and maximum of the singularity exponent $\alpha$, which acts as an appropriate weight in the reciprocal space. In fact, $\alpha_{\min} = \lim_{N' \to +\infty} D_q$ and $\alpha_{\max} = \lim_{N' \to -\infty} D_q$ characterize the scaling properties of the most concentrated and most rarified region of the intensity measure, respectively. The value of $\Delta\alpha = \alpha_{\min} - \alpha_{\max}$ may be used as a parameter reflecting the randomness of the intensity measure.

## 3. Light transmission spectra

Consider now a dielectric multilayer system where the Cartesian axes are chosen in such a way that the $z$-axis is parallel to the direction normal to the planes of the layers. The multilayer system is at the region $0 < z < L$, ($L$ being its size). The regions $z < 0$ and $z > L$ are considered to be filled by a transparent medium $V$ (vacuum in general). This multilayered system is formed by a quasiperiodic array of Fibonacci, Thue–Morse, and double-period types. Each dielectric layer is characterized by a thickness and refractive index $d_J$ and $n_J$, respectively ($J = A$ and $B$). The transparent medium $V$, which surrounds the multilayer system, has a refractive index $n_V$.

To calculate the light transmission rate (or transmittance) through the multilayer system, we use a transfer matrix approach for the electromagnetic fields. In this way, we consider that a $s$-polarized (TE waves) light of frequency $\omega$ is normally incident from a transparent medium $V$ with respect to the layered system. The reflectance and the transmittance coefficients are given by

$$R = \left| \frac{M_{21}}{M_{11}} \right|^2 \quad \text{and} \quad T = \left| \frac{1}{M_{11}} \right|^2 , \tag{13}$$

where $M_{ij}$ $(i, j = 1, 2)$ are the elements of the optical transfer matrix $M$, which links the amplitudes of the electromagnetic fields in the region $z < 0$ to the amplitudes of the electromagnetic fields in the region $z > L$. The details of these transfer matrices can be found elsewhere [25].

We now present some numerical calculations for the transmission rate $T$, due to a normal propagation of light waves of frequency $\omega$ in a quasiperiodic multilayer system of the types discussed here. In these examples we adopt the optical thickness of the individual layers as quarter-wavelength of the incident light wave, which is considered

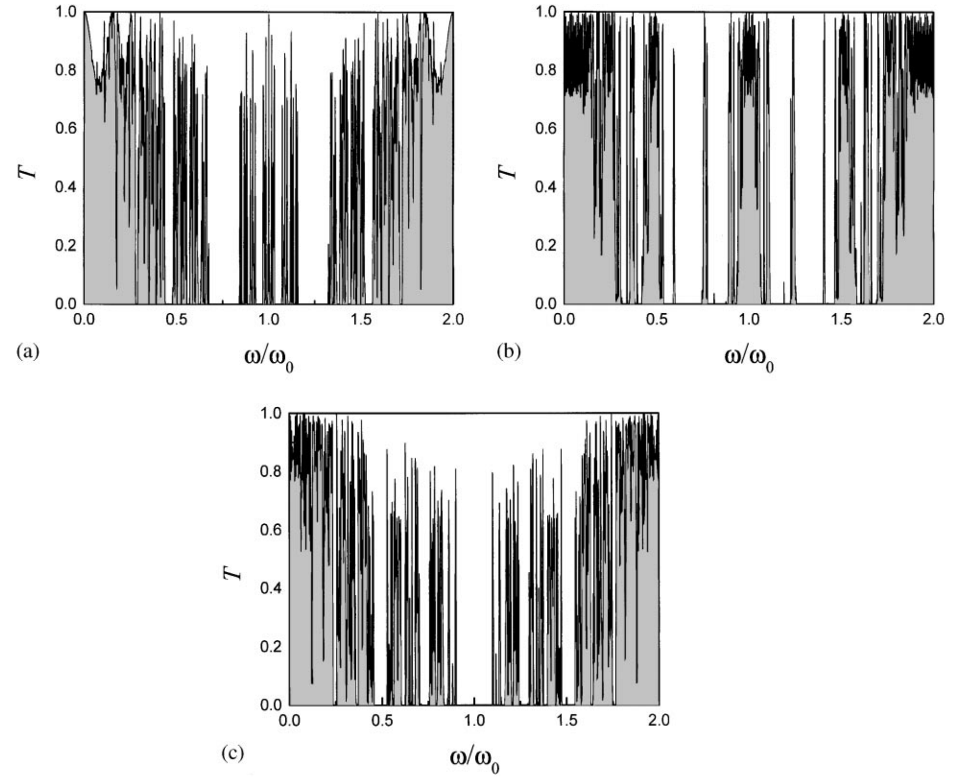


Fig. 3. Transmittance $T$ versus $\omega/\omega_0$ for the quasiperiodic structures considered here, in the interval $0.0 \leqslant \omega/\omega_0 \leqslant 2.0$. (a) Fibonacci's fifteenth generation; (b) Thue–Morse's eighth generation; (c) double-period's ninth generation.

to have a wavelength equal to $\lambda_0 = 700$ nm, i.e.: $n_A d_A = n_B d_B = \lambda_0/4$. The physical parameters used here are the same as those in Ref. [25].

Fig. 3 shows the normal-incidence transmittance spectra as a function of the reduced frequency $\omega/\omega_0$ ($\omega/\omega_0 = \lambda_0/\lambda$) in the interval $0.0 \leqslant \omega/\omega_0 \leqslant 2.0$, for the different quasiperiodic structures. The fifteenth generation (947-layers) Fibonacci multilayer system is depicted in Fig. 3a. From there, we can observe several peaks forming band gaps, symmetrically arranged around the reduced frequency $\omega/\omega_0 = 1$, which is the midgap frequency of a periodic quarter wavelength multilayer. In a previous work we have shown that this spectrum has a scaling property with respect to the generation number of the Fibonacci sequence. For example, it exhibits the same pattern at Fibonacci's seventh generation when compared to the thirteen generation in a different scale (the scale factor is equal to 26.11) [25]. Furthermore, they also exhibit self-similarity at the same generation number [26]. The optical transmission spectrum for the eighth-generation (256-layers) quasiperiodic Thue–Morse sequence, is shown in Fig. 3b. As in the Fibonacci case, the central peak is transparent and there are several symmetrical peaks forming band gaps. In Fig. 3c we show the optical transmission

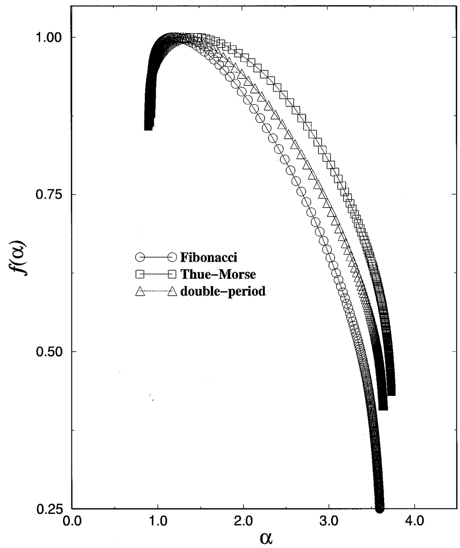


Fig. 4. The $f(\alpha)$ functions of the transmission spectra for the three quasiperiodic dielectric multilayer systems: Fibonacci, Thue–Morse and double-period.

spectrum for the ninth-generation (512-layers) double-period structure. In contrast to the others cases, we note a large gap symmetrically distributed at the incident frequency ($\omega/\omega_0 = 1$), followed by peaks near the limits of our plot. The spectrum is now completely opaque near the midgap frequency.

In order to check the multifractality aspect of the transmittance spectra described in Fig. 3, it is essential to investigate the behavior of the $f(\alpha)$ functions for the three quasiperiodic structures, using the same formalism described in the last Section. Fig. 4 illustrates the $f(\alpha)$ curves for this case. As one can see, the data points are fitted perfectly into a smooth curve, which is a characteristic of the quasiperiodic structures. Also the $f(\alpha)$ curves shrink to one point on both sides of the curves, explaining the scale invariant structures found in the transmittance spectra. Furthermore, as it was also found in the polariton case, $f(\alpha_0) = 1$, where $\alpha_0$ means the central point of the curvature.

## 4. Conclusions

We have investigated the multifractal properties of plasmon–polaritons and light waves in quasiperiodic multilayers constructed following the Fibonacci, Thue–Morse and double-period sequences. Multifractal analysis is a suitable statistical description for the study of long term dynamical behavior of a physical system. It can be completely described by the spectrum of singularities given by the $f(\alpha)$ function. Its formalism relies on the fact that the highly nonuniform probability distributions arise from the nonuniformity of the system. The multifractal analysis, presented here for both excitations, revealed a smooth $f(\alpha)$ function distributed in a finite range $[\alpha_{\min}, \alpha_{\max}]$ for all quasi-periodic structures, with a summit at $f(\alpha_0) = 1$. Our investigations demonstrated that all spectra are highly nonuniform intensity distributions, which possess scaling properties of multifractal.

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