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(A, B)-Invariance Conditions of Polyhedral Domains for Continuous-Time Systems*

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This paper provides an explicit characterisation of the (A,B)*-invariance property of polyhedral sets with respect to linear continuous-time systems. A typical application of the concept of* (A,B)*-invariance is to investigate the possibility of controlling a system subject to pointwise in time trajectory constraints. Necessary and sufficient conditions for a polyhedron to be* (A,B)*-invariant are established in the form of linear matrix relations. Some particular conditions of existence of linear state feedback laws are also presented. The study of* (A,B)*-invariance of polyhedra is then extended to the control of constrained and additively disturbed systems.*

Keywords: (A,B)-invariance; Constrained systems; Feedback control; Linear systems; Positive invariance

1. Introduction

In the framework of the geometric approach for linear systems, the concept of (A,B)-invariance has been applied to the analysis and solution of important control problems such as disturbance and input to output decoupling [1,26]. This concept has also been applied to polyhedral sets to characterise the possibility of controlling linear discrete-time systems subject to state and control constraints [3,8,12,16,18] and references therein). The main reason for considering polyhedral sets is the fact that constraints are generally pointwise in time and linear, particularly when they represent non-negativity conditions and bounds on physical variables. Another important application of (A,B)-invariance is in the solution of the persistent disturbance attenuation problem, known in the literature as the $\mathcal{L}^1$ control problem for continuous-time systems and as the ℓ^1 control problem for discrete-time systems. Indeed, it has been shown that the possibility of delivering to the system a given ℓ^1 performance is directly related to the existence of an (A,B)-invariant set [6,23] (see also [14,15] for a geometric interpretation).

The problem of controlling continuous-time linear systems subject to linear constraints has been intensively studied in recent years, mainly in the framework of the positive invariance approach (see, for example, [2,7,20,24,25] and references therein). A positively invariant domain in the state space is a domain from which the state vector trajectory cannot escape. In particular, this approach provides an algebraic test for existence of a static state-feedback control law able to guarantee the respect of the constraints. Any point in a positively invariant domain can be considered as an admissible initial point with respect to the corresponding constrained state-feedback control law. In many control engin-

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eering problems, such as motion planning in robotics, plane and missile guidance, machine setting or industrial plant operation, the control tasks can be decomposed into two successive stages: an open-loop non-linear (or a manual) stage, driving the system to a target set, and a closed-loop state-feedback stage achieving fine convergence or containment within the target set. Inclusion of the target set in the set of constraints then guarantees the respect of pointwise physical limitations and performance specifications along the system trajectory.

In many cases, however, closed-loop positive invariance of a given set, polyhedral or not, cannot be obtained under a static state-feedback. One is then led to consider more general control functions through the study of the (A,B)-invariance property. In this context, for compact polyhedra, a vertex-by-vertex characterisation of the (A,B)-invariance property with respect to uncertain systems has been proposed [4], through the study of a discrete-time approximation of the continuous-time system. The same discrete-time approximated model is used in Blanchini and Miani [5] to propose a technique to construct an approximation of the supremal (A,B)-invariant set included in a given compact polyhedron.

The main objective of this paper is to directly characterise the (A,B)-invariance property of general convex polyhedra with respect to continuous-time linear systems. Some preliminary results on cones and polyhedra are recalled in Section 2. Then, an explicit characterisation of (A,B)-invariance is obtained in Section 3 through the application of some basic results on polyhedral sets, duality in linear programming and differential analysis. The (A,B)-invariance conditions take the form of a set of linear matrix relations. A particular form of these conditions is obtained in the case of polyhedral sets which are symmetrical with respect to the origin.

In general, the polyhedron defined by physical constraints is not (A,B)-invariant. Constraint satisfaction can, however, be achieved if the set of initial states is restricted to an (A,B)-invariant set contained in the set of constraints. A natural choice to this (A,B)-invariant set is the supremal set. In Section 4, it is shown how an approximation of the supremal set can be computed.

A complementary problem is to construct a control law achieving closed-loop positive invariance for a polyhedron which satisfies the (A,B)-invariance conditions. This problem is solved in the general case by a piecewise linear control law, which is an extension to general polyhedra and continuous-time systems of the control law proposed elsewhere [3,16] for compact polyhedra. Then, a particular set of conditions, slightly more restrictive than the set of (A,B)-invariance relations, is established to characterise the existence of a linear state feedback control.

Finally, the study is extended to two important classes of systems: systems subject to linear control constraints and additively disturbed systems.

Notations and Definitions. In mathematical expressions, the symbol ':' stands for 'such that'. The components of a matrix M are noted M_{jk} and its rows M_j. By convention, inequalities between vectors and inequalities between matrices are componentwise. The absolute value $|M|$ (resp. $|v|$) of a matrix M (resp. of a vector v) is defined as the matrix (resp. vector) of the absolute value of its components. An essentially non-negative matrix M is a matrix having all its off-diagonal terms non-negative: $M_{jk} \geq 0\ \forall k \neq j$. The cardinality of a set I, card(I), is defined as the number of elements in I.

2. Cones and Polyhedra

Some fundamental concepts related to sets of linear spaces are first recalled.

Let S be a set in a normed linear space $\mathcal{X}$, a norm being represented by the symbol $\|.\|$. The set S is bounded if there exists a scalar $s > 0$ such that $\|x\| \leq s$, $\forall x \in S$. S is closed if it contains all of its closure points. S is compact if it is bounded and closed.

All the sets studied in this paper are closed sets.

2.1. Polyhedral Sets

A convex polyhedron in $\Re^n$, $R[G, \rho]$, with $G \in \Re^{g \times n}$ and $\rho \in \Re^g$, is defined by the system of linear inequalities

$$R[G, \rho] = \{x \in \Re^n: Gx \leq \rho\}$$

A polyhedral cone in $\Re^n$, $R[G,0]$, is defined by the system of linear inequalities

$$R[G, 0] = \{x \in \Re^n: Gx \leq 0\}$$

A symmetrical polyhedron in $\Re^n$, $S(Q, \phi)$, is defined, for $\phi \leq 0$, by the system of linear inequalities

$$S(Q, \phi) = \{x \in \Re^n: |Qx| \leq \phi\}$$

2.2. Generators of Polyhedral Cones

A set of generators of the polyhedral cone $R[G, 0]$ is defined as follows.

Definition 2.1. The column-vectors of matrix M form a set of generators of the polyhedral cone $R[G, 0]$ if and only if there exists a non-negative vector ξ such that $x = M\xi$, $\forall x \in R[G, 0]$.

Definition 2.2. A set of generators M of $R[G, 0]$ is called a minimal generating set if it has the smallest number of vectors.

The affine hull (or lineality space) of $R[G, 0]$ is defined by [21]: $\mathcal{A} = R[G, 0] \cap -R[G, 0] = \{x \in \Re^n\colon Gx = 0\}$. The dimension of the affine hull is $h = n-\text{rank}(G)$. In the general case, any polyhedral cone $R[G, 0]$ can be decomposed into the form: $R[G, 0] = P + \mathcal{A}$, where P is a proper cone [21]. If $\mathcal{A} = \{0\}$, the cone is pointed and a set of generators is obtained by selecting one non-zero vector of each extremal ray of the cone. This set of vectors forms a minimal generating set of $R[G, 0]$ [9].

2.3. Decomposition of Polyhedra

The geometric structure of a polyhedron $R[G, \rho] \subset \Re^n$ is often described in terms of faces and facets. The following definitions are taken from Schrijver [21].

Definition 2.3. A subset F of $R[G, \rho]$ is a face if there exists a row-vector $c \in \Re^n$ such that F is the set of vectors x attaining $(\max\{cx\colon Gx \le \rho\})$, provided that this maximum is finite.

Definition 2.4. A minimal face of $R[G, \rho]$ is a face not containing any other face.

Definition 2.5. A facet of $R[G, \rho]$ is a maximal face, that is, a face not contained in any other face of $R[G, \rho]$.

Any polyhedron $R[G, \rho] \subset \Re^n$ admits a minimal decomposition (see, for example, Schrijver [21]) as the sum of the cone $R[G, 0]$, which is called its characteristic cone, and of a polytope, Π, defined by its vertices $(x_1, \ldots, x_p)$:

$$\forall x \in R[G, \rho], \exists\, y \in R[G, 0], z \in \Pi\colon x = y + z \qquad (1)$$

Each vertex $(x_1, \ldots, x_p)$ of the polytope Π is on a minimal face of the polyhedron $R[G, \rho]$.

If the vectors M_j ($j = 1, \ldots, q$) form a set of generators of the polyhedral cone $R[G, 0]$, then any point $x \in R[G, \rho]$ can be defined by the set of parameters $(\alpha_1, \ldots, \alpha_q)$ and $(\beta_1, \ldots, \beta_p)$ through the linear expression

$$x = \sum_{j=1}^{q} \alpha_j M_j + \sum_{i=1}^{p} \beta_i x_i \qquad (2)$$

with

$$\alpha_j \ge 0\ \forall j = 1, \ldots, q;$$
$$0 \le \beta_i \le 1\ \forall i = 1, \ldots, p; \sum_{i=1}^{p} \beta_i \le 1$$

Note that the polytope Π always belongs to the polyhedron $R[G, \rho]$, as it can be shown by selecting $y = 0$ in decomposition (1). On the contrary, the characteristic cone $R[G, 0]$ belongs to the polyhedron $R[G, \rho]$ if and only if the zero vector belongs to $R[G, \rho]$, that is, if all elements of vector ρ are non-negative. This condition will be assumed in the sequel.

For $\rho \ge 0$, a partition of $R[G, \rho]$ can be derived from parametrisation (2). This partition is an extension to the general case of the partition proposed by Gutman and Cwikel [16] and Blanchini [3] for compact convex polyhedra. Each region $\mathcal{X}_r$ of $R[G, \rho]$ is generated through relation (2) by a set of generators and/or vertices $(M_{j'}, x_{i'})$, $j' \in \mathcal{J}_r$, $i' \in \mathcal{I}_r$, such that

- $\text{card}(\mathcal{J}_r) + \text{card}(\mathcal{I}_r) = n$.
- A point $x \in \mathcal{X}_r$ is given by

$$x = \sum_{j' \in \mathcal{J}_r} \alpha_{j'} M_{j'} + \sum_{i' \in \mathcal{I}_r} \beta_{i'} x_{i'} \qquad (3)$$

with

$$\alpha_{j'} \ge 0; 0 \le \beta_{i'} \le 1; \sum_{i' \in \mathcal{I}_r} \beta_{i'} \le 1$$

The transition between two adjacent regions is characterised by a pivoting operation in which one of the coefficients $(\alpha_{j'}, \beta_{i'})$ becomes null and either a generator M_j, $j \notin \mathcal{J}_r$ or a vertex x_i, $i \notin \mathcal{I}_r$, replaces, in representations (3), the generator or vertex for which $\alpha_{j'}$ or $\beta_{i'}$ has become null. The intersection of two adjacent regions has an empty interior, and the union of all regions $\mathcal{X}_r$ is the polyhedron $R[G, \rho]$.

An example of decomposition of a polyhedron is given in Fig. 1.

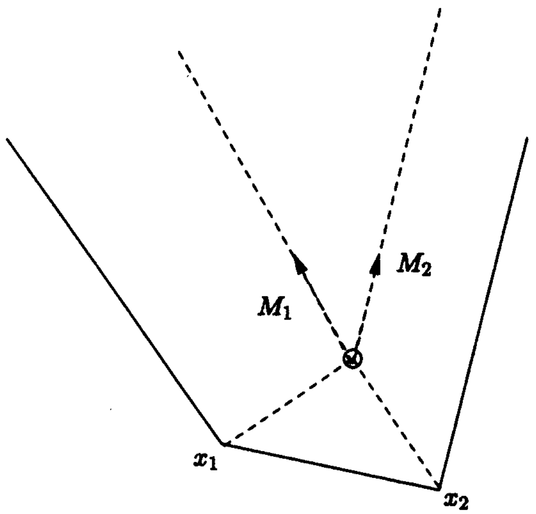

Fig. 1. A polyhedron $R[G, \rho]$ and its associated regions.

2.4. Farkas' Lemma

The following form of Farkas' lemma will be used in this study (see, for example, Schrijver [21]).

Lemma 2.1. Let M be a matrix and v a vector. Then, $\exists x : Mx \leq v$ if and only if $yv \geq 0\ \forall y \geq 0 : yM = 0$.

The set of candidate row-vectors $y \geq 0$ such that $yM = 0$ forms a pointed polyhedral cone. In this study, this cone is called the non-negative left kernel of matrix M.

Let W be a non-negative matrix whose rows form a minimal generating set of the non-negative left kernel of M. Then, a statement equivalent to Lemma 2.1 is (see also de Santis [9])

$$\exists x : Mx \leq v \text{ if and only if } Wv \geq 0. \tag{4}$$

As shown in Keerthi and Gilbert [18], it is possible to compute matrix W by the Fourier–Motzkin elimination technique (see also Schrijver [21]).

In the case of equality constraints, the following extended version of Farkas' lemma can be stated.

Lemma 2.2. Let M and V be two matrices of appropriate dimensions. Then, $\exists X : MX = V$ if and only if $yV = 0\ \forall y : yM = 0$.

3. (A,B)-Invariance of Polyhedra

Consider the linear continuous-time system:

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{5}$$

with $t \geq 0$, where $x \in \Re^n$ is the state vector and $u \in \Re^m$ is the control vector. An (A,B)-invariant set of system (5) is defined as follows.

Definition 3.1. A set $S \subset \Re^n$ is said to be (A,B)-invariant (or controlled invariant) with respect to system (5) if for all initial state $x(0) \in S$ there exists a control function $u(t)$, $t \geq 0$, such that $x(t) \in S$, $\forall t \geq 0$.

The class C_L of control functions is defined as follows.

Definition 3.2. The control function $u(x,t)$ is said to belong to the class C_L if it is continuous and Lipschitz with respect to the state vector.

Consider now the state x at time t and the power series:

$$x(t + \Delta t) = x(t) + \dot{x}(t)\Delta t + \ldots + \frac{x^{(k)}}{k!}\Delta t^k + \ldots$$

Throughout the paper, the term infinitesimal motion will be used to mean the linear part of the power series of $x(t + \Delta t)$, that is, $x(t) + \dot{x}(t)\Delta t$. This reflects the fact that for small Δt the higher-order terms of the power series are negligible.

The following proposition has been shown in a slightly different framework in Seifert [22], Corollary 1, p. 295.

Proposition 3.1. A necessary and sufficient condition for (A,B)-invariance of a convex set S with respect to system (5), with a control function in the class C_L, is the existence of a control function in the class C_L defined on the boundary of S, for which, at any point x_b of this boundary, the infinitesimal motion starting at x_b remains in S.

In the sequel, the study of (A,B)-invariance will be focused on convex polyhedral sets of $\Re^n$ containing the 0-vector

$$S = R[G, \rho] = \{x : Gx \leq \rho\}$$

Let T be a matrix whose rows form a minimal generating set of the non-negative left kernel of the matrix product GB, denoted Γ and defined by

$$\Gamma = \{w \in \Re^g : w \geq 0, w^T(GB) = 0\} \tag{6}$$

Clearly, this cone is pointed, because $\Gamma \subset \Re^g_+$. Therefore, as mentioned in Section 2.2, a minimal generating set of Γ can be obtained by selecting one non-zero vector of each extremal ray of Γ.

Theorem 3.1. Let T be a matrix whose rows form a minimal generating set of the polyhedral cone Γ (6). A polyhedral set $R[G, \rho] \subset \Re^n$ is (A,B)-invariant with respect to system (5) if and only if there exists a matrix Y such that

$$YG = TGA \tag{7}$$

$$Y\rho \leq 0 \tag{8}$$

$$Y_{ij} \geq 0 \text{ if } T_{ij} = 0 \tag{9}$$

Proof.
Necessity. Assume (A,B)-invariance of $R[G, \rho]$ with respect to system (5). Consider a row T_i of T. From the definition of matrix T, vector T_i is a generator of the cone Γ. It has non-negative components and satisfies:

$$T_iGB = 0 \tag{10}$$

The following linear programme, denoted (P_i), can be associated to this vector:

$$\max_x z_i = T_iGAx$$

$$\text{subject to } G_jx = \rho_j \quad \text{if } T_{ij} \neq 0 \tag{11}$$

$$G_jx \leq \rho_j \quad \text{if } T_{ij} = 0 \tag{12}$$

The dual of problem (P_i), denoted (D_i), is defined as follows:

$$\min_{Y_i} y_i = Y_i\rho$$

$$\text{subject to: } Y_iG = T_iGA \tag{13}$$

$$Y_{ij} \geq 0 \text{ if } T_{ij} = 0 \tag{14}$$

Then, one of the two following situations happens:

(a) The set of conditions (11), (12) is consistent. It then defines a face of $R[G, \rho]$. (A,B)-invariance of $R[G, \rho]$ requires, at each point x of this face, admissibility of the infinitesimal motion, and thus the existence of a control vector u such that

$$G_j\dot{x} = G_jAx + G_jBu \leq 0 \; \forall j : T_{ij} \neq 0 \tag{15}$$

Condition (10) implies

$$\sum_{j:\, T_{ij}\neq 0} T_{ij}G_jB = 0$$

Since $T_{ij} \geq 0$, left-multiplication of each condition (15) by T_{ij} yields, for all x satisfying (11), (12):

$$z_i = T_iGAx = \sum_{j:T_{ij}\neq 0} T_{ij}G_jAx \leq 0$$

Thus, the optimal solution of problem (P_i), denoted z_i^* satisfies $z_i^* \leq 0$. Then, the optimal criterion of the dual problem (D_i), y_i^*, satisfies

$$y_i^* = z_i^* = Y_i^* \rho \leq 0$$

(b) The set of conditions (11), (12) is not consistent, the primal problem (P_i) is thus infeasible, but since $R[G, \rho]$ is not empty, a relaxed version of (P_i) can be made feasible by replacing some equality constraints $G_jx = \rho_j$ by the associated inequality constraints $G_jx \leq \rho_j$. A solution to the relaxed dual problem is then feasible for the original dual problem. Therefore, the optimal solution of the dual problem (D_i) is unbounded. Thus, there exists a row-vector Y_i satisfying (13), (14) and such that $Y_i\rho \leq 0$.

The same argument can be applied to all the rows of matrix T, showing the necessity of conditions (7), (8), (9).

Sufficiency. Suppose the existence of a matrix Y satisfying conditions (7), (8), (9). Consider a point x on the boundary of $R[G, \rho]$. The rows of matrix G and vector ρ can be partitioned into two subsets, with indices 1, 2, and dimensions g_1, g_2, and reordered as

$$G = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix}, \; \rho = \begin{bmatrix} \rho_1 \\ \rho_2 \end{bmatrix}$$

so that x satisfies:

$$\begin{cases} G_1x = \rho_1 \\ G_2x < \rho_2 \end{cases} \tag{16}$$

Let T_b be a non-negative matrix whose rows form a minimal generating set of the polyhedral cone $\{w \geq 0 : w^T(G_1B) = 0\}$. Any row vector T_{bi} can be complemented to generate a vector $t_i \in \Gamma$ defined by: $t_i = [T_{bi} \; 0]$. Then, by definition of matrix T, $\exists \xi_i \geq 0$ such that $t_i = \xi_iT$. Accordingly, all the rows of matrix $t = [T_b \; 0]$ belong to Γ, and thus, $\exists \; \Xi \geq 0$ such that

$$t = \Xi T$$

Left-multiplication of relations (7), (8), by Ξ yields, with $Z = \Xi Y$:

$$ZG = tGA = T_bG_1A \tag{17}$$

$$Z\rho \leq 0 \tag{18}$$

Furthermore, conditions

$$\begin{cases} t_{ij} = \Sigma_k\xi_{ik}T_{kj} \\ Z_{ij} \; \Sigma_k\xi_{ik}Y_{kj} \end{cases}$$

and $\Xi \geq 0$ show that $t_{ij} = 0$ if, for each k, either $\xi_{ik} = 0$ or $T_{kj} = 0$. Using condition (9), this implies, respectively, either $\xi_{ik} = 0$ or $Y_{kj} \geq 0$, and thus

$$Z_{ij} \geq 0 \text{ if } t_{ij} = 0 \tag{19}$$

The three conditions (17), (18), (19) show, by duality, that, for x satisfying (16):

$$T_b G_1 A x \leq 0 \tag{20}$$

By application of Lemma 1 in the form (4), condition (20) is equivalent to the existence of a vector $u \in \Re^m$ such that

$$G_1 \dot{x} = G_1 A x + G_1 B u \leq 0 \tag{21}$$

Now, recall the decomposition of polyhedron $R[G, \rho]$ as the sum of the characteristic cone and a polytope Π (1). Relation (21) applies to all the points on the boundary of $R[G, \rho]$, in particular to all the vertices $(x_1, \ldots, x_p)$ of the polytope Π. A set of admissible controls $(v_1, \ldots, v_p)$ can be associated to this set of vertices. They satisfy: $G_k A x_i + G_k B v_i \leq 0 \; \forall \; k$: $G_k x_i = \rho_k$. Similarly, the characteristic cone $R[G,0]$ trivially satisfies relations (7), (8), (9) (with $\rho = 0$, $Y = 0$). Therefore, a set of controls $(w_1, \ldots, w_q)$ can be associated to the set of generators M_j of $R[G, 0]$, so as to satisfy: $G_k A M_j + G_k B w_j \leq 0 \; \forall \; k : G_k M_j = 0$.

Each point of $R[G, \rho]$ being represented by the set of coordinates $(\alpha_1, \ldots, \alpha_q, \beta_1, \ldots, \beta_p)$ through relation (2), the following control function can be considered:

$$u(x) = \sum_{j=1}^{q} \alpha_j w_j + \sum_{i=1}^{p} \beta_i v_i \tag{22}$$

with

$$\alpha_i \geq 0 \; \forall j, \; 0 \leq \beta_i \leq 1 \; \forall i, \; \sum_{i=1}^{p} \beta_i \leq 1$$

Using the set of coordinates derived from the partition of $R[G, \rho]$ proposed in Section 2.3 (with the assumption $\rho \geq 0$), it can be readily shown that the control function (22) is defined at any point of $R[G, \rho]$, is continuous and Lipschitz. It generates a feasible motion from each point of the boundary of $R[G, \rho]$. Thus, using Proposition 1, the polyhedron $R[G, \rho]$ is (A,B)-invariant with respect to system (5), with a control function in the class C_L. □

Remarks.

- In the particular case when T reduces to the null row-vector, the set of conditions (7)–(9) is trivially satisfied with Y equal to the null row-vector. The polyhedral set $R[G, \rho]$ is then trivially (A,B)-invariant.
- In the case of an autonomous system ($B = 0$), Γ (6) is the whole non-negative orthant $\Re_+^g$, $T = I_g$, and relations (7)–(9) reduce to the classical positive invariant relations for autonomous systems [2,7,25].
- The control law (22) can be seen as an extension to general polyhedra and continuous-time systems of the control law proposed elsewhere [3,16] for compact polyhedra with respect to discrete-time systems.
- From relations (7)–(9), it can be seen that (A,B)-invariance of the characteristic cone is a necessary condition for (A,B)-invariance of $R[G, \rho]$.

Theorem 3.1 can be specialised to the case of symmetrical polyhedra $S(Q, \phi)$. Consider a matrix $[T_1 \; T_2]$ whose rows form a minimal generating set of the polyhedral cone Γ (6), with

$$G = \begin{bmatrix} Q \\ -Q \end{bmatrix}$$

Now, form the matrix $\mathcal{T}$ as a submatrix of $T_1 - T_2$, obtained by deleting the rows $T_{1i} - T_{2i}$ for which either $T_{1i} - T_{2i} = 0$ or $T_{1i} - T_{2i} = -T_{1j} + T_{2j}$ for some $j < i$. The following result can be established.

Corollary 3.1. A symmetrical polyhedral set $S(Q, \phi) \subset \Re^n$ is (A,B)-invariant with respect to system (5) if and only if there exists a matrix Y such that

$$YQ = \mathcal{T} QA \tag{23}$$

$$\tilde{Y}\phi \leq 0 \tag{24}$$

where $\tilde{Y}$ is given by

$$\tilde{Y}_{ij} = \begin{cases} |Y_{ij}| & \text{if } \mathcal{T}_{ij} = 0 \\ Y_{ij} & \text{if } \mathcal{T}_{ij} > 0 \\ -Y_{ij} & \text{if } \mathcal{T}_{ij} < 0 \end{cases}$$

Proof: From Theorem 3.1, $S(Q, \phi)$ is (A,B)-invariant if and only if there exist non-negative matrices Y_1 and Y_2 such that

$$(Y_1 - Y_2)Q = (T_1 - T_2)QA \tag{25}$$

$$(Y_1 + Y_2)\phi \leq 0 \tag{26}$$

$$Y_{1ij} \geq 0 \text{ if } T_{1ij} = 0, \; Y_{2ij} \geq 0 \text{ if } T_{2ij} = 0 \tag{27}$$

The rows i for which $T_{1i} - T_{2i} = 0$ do not need to be considered because in this case relations (25), (26) are trivially verified with $Y_{1i} = Y_{2i} = 0$. The same applies to the rows i for which $T_{1i} - T_{2i} = -T_{1j} + T_{2j}$ for some $j < i$, because if for the row j

there exist row vectors Y_{1j} and Y_{2j} such that (25), (26) are verified, then the same relations are verified for the row i with $Y_{1i} = Y_{2j}$ and $Y_{2i} = Y_{1j}$.

Considering now the matrix $\mathcal{T}$, the following hold (otherwise the corresponding generators in $[T_1\ T_2]$ would not belong to the minimal generating set):

- $\mathcal{T}_{ij} = 0$ only if the corresponding elements in $\mathcal{T}_1$ and $\mathcal{T}_2$ are also null.
- $\mathcal{T}_{ij} > 0$ (< 0) only if the corresponding elements in T_1 and T_2 are respectively positive (null) and null (positive).

In view of these facts, from relations (25), (26), controlled invariance of $S(Q, \phi)$ is equivalent to the existence of non-negative matrices $\mathcal{Y}_1$ and $\mathcal{Y}_2$ verifying

$$(\mathcal{Y}_1 - \mathcal{Y}_2)Q = \mathcal{T}QA \tag{28}$$

$$(\mathcal{Y}_1 + \mathcal{Y}_2)\phi \leq 0 \tag{29}$$

$$\mathcal{Y}_{1ij} \geq 0 \text{ if } \mathcal{T}_{ij} = 0 \text{ or } \mathcal{T}_{ij} < 0 \tag{30}$$

$$\mathcal{Y}_{2ij} \geq 0 \text{ if } \mathcal{T}_{ij} = 0 \text{ or } \mathcal{T}_{ij} > 0 \tag{31}$$

Now, let $Y = \mathcal{Y}_1 - \mathcal{Y}_2$ and consider the matrices Y^+ and Y^- defined by

$$Y^+_{ij} = \begin{cases} \max\{Y_{ij}, 0\} & \text{if } \mathcal{T}_{ij} = 0 \\ Y_{ij} & \text{if } \mathcal{T}_{ij} > 0 \\ 0 & \text{if } \mathcal{T}_{ij} < 0 \end{cases} \tag{32}$$

$$Y^-_{ij} = \begin{cases} \max\{-Y_{ij}, 0\} & \text{if } \mathcal{T}_{ij} = 0 \\ 0 & \text{if } \mathcal{T}_{ij} > 0 \\ -Y_{ij} & \text{if } \mathcal{T}_{ij} < 0 \end{cases}$$

Necessity is proven by observing that these matrices are such that $Y^+ - Y^- = Y = \mathcal{Y}_1 - \mathcal{Y}_2$ and $(Y^+ + Y^-)\phi = \tilde{Y}\phi \leq (\mathcal{Y}_1 + \mathcal{Y}_2)\phi \leq 0$. Sufficiency follows from the fact that Y^+ and Y^- verify relations (28)–(31). □

Very often, the desired property is not only (A,B)-invariance, but also convergence to the origin with a prescribed rate. Consider then the function

$$\Psi(x) = \max_k \left\{\frac{G_k}{\rho_k} x\right\} \tag{33}$$

For compact polyhedra containing the origin in its interior, $\rho > 0$, $\Psi(x)$ is the Minkowski functional of $R[G, \rho]$ [19]. It can be shown that in this case $\Psi(x)$ is positive definite and continuous.

The total derivative of $\Psi(x)$ with respect to system (5) is given by

$$\mathcal{D}^+(x,u) = \lim_{\Delta t \to 0^+} \sup \left\{\frac{\Psi(x + \Delta t(Ax+Bu)) - \Psi(x)}{\Delta t}\right\}$$

Definition 3.3. A compact polyhedral set $R[G, \rho] \subset \Re^n$ is said to be (A,B)-invariant with an exponential convergence rate ϵ, $\epsilon > 0$, if for all initial state $x(0) \in R[G, \rho]$ there exists a control function $u(t)$, $t \geq 0$, such that the trajectory of the state vector verifies

$$\mathcal{D}^+(x,u) \leq -\epsilon\Psi(x)$$

Theorem 3.2. Let T be a matrix whose rows form a minimal generating set of the polyhedral cone Γ (6). A compact polyhedral set $R[G, \rho] \subset \Re^n$ is (A,B)-invariant with respect to system (5), with an exponential convergence rate, ϵ, if and only if there exists a matrix Y such that

$$YG = TGA \tag{34}$$

$$Y\rho \leq -\epsilon T\rho \tag{35}$$

$$Y_{ij} \geq 0 \text{ if } T_{ij} = 0 \tag{36}$$

Proof. The proof follows the same lines of the proof of Theorem 3.1 and is only outlined. Let $I(x)$ denote the set of indices k for which $\Psi(x) = (G_k)/(\rho_k)x$. By definition of $\Psi(x)$,

$$\begin{cases} G_k x = \rho_k \Psi(x) & \text{for } k \in I(x) \\ G_l x < \rho_l \Psi(x) & \text{for } l \notin I(x) \end{cases}$$

Furthermore, by virtue of the continuity of $\Psi(x)$, it can be shown that [25]

$$\mathcal{D}^+(x,u) = \max_{k \in I(x)} \left\{\frac{G_k(Ax+Bu)}{\rho_k}\right\} \tag{37}$$

Now, consider the matrix $T_b = [T_{b_k}\ 0]$, where the rows of T_{b_k} form a set of generators of the polyhedral cone $\{w \geq 0\colon w^T(G_k B) = 0\}$. Then, from (34)–(36) and the proof of Theorem 3.1, there exists a matrix Y_b, with $Y_{bij} \geq 0$ if $T_{bij} = 0$, such that $T_b GAx = Y_b Gx \leq Y_b\rho\Psi(x) \leq -\epsilon T_b \rho\Psi(x)$. Therefore, from Lemma 2.1, there exists a control vector u such that $G_k Ax + G_k Bu \leq -\epsilon\rho_k\Psi(x)$ and thus, $\mathcal{D}^+(x,u) \leq -\epsilon\Psi(x)$. □

Note that in this case $\Psi(x)$ is a Lyapunov function of system (5).

4. The Supremal (A,B)-Invariant Set

Suppose now that, as a design specification resulting from physical limitations in the plant to be controlled, the state of system (5) is constrained to evolve inside a convex set S. In general, a given set S is not (A,B)-invariant. A possible solution to this problem is therefore to restrict the initial state to an (A,B)-invariant set contained in S. Furthermore, it is desirable that such a set should be as large as possible. It turns out that a supremal set exists, which results from the following property, whose proof is straightforward.

Lemma 4.1. The family of all (A,B)-invariant sets contained in a closed convex set S is closed under the operation 'convex hull of the union'.

Since S is assumed closed, this lemma guarantees the existence in the family of (A,B)-invariant sets contained in S of a supremal member (a member which contains all the other members):

$$C^{\infty}(S) \stackrel{\triangle}{=} \text{supremal } (A,B)\text{-invariant set contained in } S$$

In the polyhedral case, $S = R[G, \rho]$, and for discrete-time systems, iterative formulas are available to exactly compute $C^{\infty}(S)$ up to a given precision [3,11,17]. And it turns out that in many cases such a supremal set is polyhedral. This is not true for continuous-time systems, and the exact computation of $C^{\infty}(S)$ in this case becomes very difficult. As shown elsewhere [4,5], arbitrarily close polyhedral approximations of $C^{\infty}(S)$ can nevertheless be computed, if one considers the Euler Approximating System (EAS):

$$x(k+1) = (I+\tau A)x(k)+\tau Bu(k),\ \tau > 0 \qquad (38)$$

Such approximations are based on the following result [4,5].

Proposition 4.1. If $R[G, \rho]$ is (A,B)-invariant with respect to the EAS (38), then $R[G, \rho]$ is (A,B)-invariant with respect to system (5).

Arbitrarily close polyhedral approximations of $C^{\infty}(S)$ can therefore be obtained by computing the supremal (A,B)-invariant set contained in $R[G, \rho]$, with respect to the discrete-time system (38), with τ sufficiently small.

5. Computation of a Control Law

The satisfaction of the (A,B)-invariance relations presented beforehand guarantees the existence of a continuous and Lipschitz control function which forces the trajectory of the state to belong to the (A,B)-invariant polyhedron. That does not presuppose, however, a particular type of control law. A closed-loop continuous and piecewise linear control law has been proposed in Blanchini [4], but for compact polyhedra only. The extension of such a law to the non-compact case, presented in the proof of Theorem 3.1, is now detailed.

Consider the partition of $R[G, \rho]$ given by relation (3), and the control function (22). Let X_r be a square matrix whose columns are the generators/vertices defining the region $\mathcal{X}_r$, and let U_r be a matrix whose columns are the control vectors w_j/v_i associated to such generators/vertices. A continuous and piecewise linear state feedback control law is then given by

$$u(t) = F_r x(t) = U_r(X_r)^{-1}x(t), \text{ for } x(t) \in \mathcal{X}_r \qquad (39)$$

This control law is a possible implementation of the law (22).

Note that compact polyhedra are completely represented by their vertices, because their characteristic cone is the null vector. The regions into which a compact polyhedron is divided are the compact polyhedra defined by the convex hull of the origin and n vertices (n being the dimension of the system). In this case, the control law (39) is exactly that proposed in Blanchini [4].

If the polyhedron $R[G, \rho]$ has to be partitioned into many regions, the implementation of the control law (39) can become very difficult. It has been observed that in most cases closed-loop positive invariance can also be obtained using much simpler state-feedback laws, which are simply linear. The linear state-feedback case is characterised by (A,B)-invariance conditions which are slightly stronger than those of Theorem 3.1. This is shown in the following theorem, whose proof can be found in [13].

Theorem 5.1. Let the rows of matrix T form a set of generators of the polyhedral cone Γ (6) and the rows of matrix

$$\begin{bmatrix} T \\ N \end{bmatrix}$$

span the left kernel of the matrix product GB. If there exists an essentially non-negative matrix H such that

$$\begin{bmatrix} T \\ N \end{bmatrix} HG = \begin{bmatrix} T \\ N \end{bmatrix} GA \qquad (40)$$

$$TH\rho \leq 0 \qquad (41)$$

then the polyhedral set $R[G, \rho] \subset \Re^n$ is (A,B)-invariant with respect to system (5) and positively invariant under a linear state feedback law:

$$u = Fx + u_c \qquad (42)$$

In practice, the existence of an admissible control law (42) can be directly tested, and the solution constructed by linear programming, as in Vassilaki and Bitsoris [25].

If the constant control term u_c is not null, the trajectory of the closed-loop system does not converge to the zero state. Assuming $R[G, \rho]$ compact, convergence to the origin can nevertheless be obtained under the continuous and piecewise linear control law: $u = Fx + u_c\Psi(x)$, where $\Psi(x)$ is the Minkowski functional of $R[G, \rho]$ (33) [13]. A discontinuous variable structure control with the same structure in terms of number of sectors as this law was proposed in Blanchini and Miani [5] for single input systems.

6. Extensions

6.1. Systems Subject to Control Constraints

Suppose now that the control entries of system (5) are subject to the following constraints:

$$u(t) \in \mathcal{U} \subset \mathcal{R}^m, \forall t \geq 0 \tag{43}$$

This type of constraint is frequently found in practical applications, being generally associated to physical limitations on the actuators.

Definition 6.1. A set $S \subset \Re^n$ is said to be $\mathcal{U}$-(A,B)-invariant with respect to system (5), (43) if for all initial state $x(0) \in S$ there exists a control function $u(t)$, with $u(t) \in \mathcal{U}\ \forall t \geq 0$, such that $x(t) \in S\ \forall t \geq 0$.

Consider now the polyhedral case: $S = R[G, \rho]$, $\mathcal{U} = R[U, \psi] = \{u \in \Re^m : Uu \leq \psi\}$. Let the rows of matrix $[T_g\ T_u]$ form a minimal generating set of the polyhedral cone Γ_u defined by

$$\Gamma_u = \left\{ \begin{bmatrix} w_g \\ w_u \end{bmatrix} : \begin{bmatrix} w_g \\ w_u \end{bmatrix} \geq 0, \quad [w_g^T\ w_u^T] \begin{bmatrix} GB \\ U \end{bmatrix} = 0, \right\} \tag{44}$$

Theorem 6.1. Let $[T_g\ T_u]$ be a matrix whose rows form a minimal generating set of the polyhedral cone Γ_u (44). A polyhedral set $R[G, \rho] \subset \Re^n$ is $\mathcal{U}$-(A,B)-invariant with respect to system (5), (43), with $\mathcal{U} = R[U, \psi]$, if and only if there exists a matrix Y such that

$$YG = T_gGA \tag{45}$$

$$Y\rho \leq T_u\psi \tag{46}$$

$$Y_{ij} \geq 0 \text{ if } T_{g_{ij}} = 0 \tag{47}$$

The proof of this theorem can be found in Dórea [10]. It essentially follows the same lines as the proof of Theorem 3.1.

6.2. Systems Subject to Bounded Additive Disturbances

Consider the following linear continuous-time system:

$$\dot{x}(t) = Ax(t) + Bu(t) + Ed(t) \tag{48}$$

where $d \in \Re^q$ is a disturbance vector, supposed constrained to evolve inside a bounded domain $\mathcal{D} \subset \Re^q$:

$$d(t) \in \mathcal{D}\ \forall t \geq 0 \tag{49}$$

One can notice that this kind of disturbance acts continuously in time, and its energy is infinite. This is why it is named by some authors persistent disturbances.

Definition 6.2. A set $S \subset \Re^n$ is said to be $\mathcal{D}$-(A,B)-invariant with respect to system (48), (49) if for all initial state $x(0) \in S$ there exists a control function $u(t)$, $t \geq 0$, such that $x(t) \in S\ \forall d(t) \in \mathcal{D}$, $\forall t \geq 0$.

This definition assumes that the disturbance vector is not measured.

Consider now the polyhedral case: $S = R[G, \rho]$, $\mathcal{D} = R[D, \omega] = \{d \in \Re^q : Dd \leq \omega\}$. Define the components δ_i of vector δ as follows:

$$\delta_i = \max_{d \in R[D,\omega]} G_iEd$$

Theorem 6.2. Let T be a matrix whose rows form a minimal generating set of the polyhedral cone Γ (6). A polyhedral set $R[G, \rho] \subset \Re^n$ is $\mathcal{D}$-(A,B)-invariant with respect to system (5), with $\mathcal{D} = R[D, \omega]$, if and only if there exists a matrix Y such that

$$YG = TGA \tag{50}$$

$$Y\rho \leq -T\delta \tag{51}$$

$$Y_{ij} \geq 0 \text{ if } T_{ij} = 0 \tag{52}$$

The proof of this theorem follows also the same lines as the proof of Theorem 3.1 [10]. One only has to notice that the role of vector δ is to absorb the effect of the disturbances.

A practical application of $\mathcal{D}$-(A,B)-invariant polyhedra is in the solution of the persistent disturbance attenuation problem, also known as the $\mathcal{L}^1$ control problem. Indeed, suppose that the system is excited by bounded disturbances, satisfying: $|d_i(t)| \leq 1\ \forall i,\ \forall t \geq 0$, and that the control goal is to limit the maximal amplitude of the output vector given by

$$y(t) = Cx(t)$$

It can be shown that a level of attenuation, say γ, is achievable only if there exists a $\mathcal{D}$-(A,B)-invariant set contained in the polyhedron $\{x : |C_j x| \leq \gamma, \forall j\}$ (see, for example, [6,23]). Then, theoretically, the optimal attenuation level γ^* is obtained as the smallest scalar for which the supremal $\mathcal{D}$-(A,B)-invariant set is not empty. In practice, a decreasing sequence of achievable values of γ can be computed to approximate the minimal feasible value, γ^*. At each iteration, a polyhedral approximation of the supremal $\mathcal{D}$-(A,B)-invariant set is computed and the current value of γ is proved feasible as long as this set is not empty.

7. Numerical Example

Consider the system (5) for which

$$A = \begin{bmatrix} -0.14 & -1.27 \\ -1.35 & 0.98 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Suppose that the state vector is subject to the constraint $x(t) \in R[G, \rho]\ \forall t \geq 0$, with

$$G = \begin{bmatrix} 0.2 & 0.2 \\ -1 & -1 \\ -1 & 0.35 \\ 0.25 & -0.5 \end{bmatrix}, \rho = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

and that convergence with an exponential rate $\epsilon = 0.5$ is desired.

$R[G, \rho]$ is not (A,B)-invariant, but the constraint $x(t) \in R[G, \rho]$ can be respected if the initial state is restricted to an (A,B)-invariant set contained in $R[G, \rho]$. Such a set can be obtained through the computation of an approximation of $C^\infty(R[G, \rho], \epsilon)$, the supremal (A,B)-invariant set with exponential convergence rate ϵ in $R[G, \rho]$.

Following the procedure described in Section 4, the (A,B)-invariant set $R[G^F, \rho^F]$ is computed with a precision of 10^{-4}, where

$$G^F = \begin{bmatrix} G \\ \hline -86.4109 & 11.1300 \\ 86.4109 & -11.1300 \end{bmatrix}, \rho^F = \begin{bmatrix} \rho \\ \hline 64.8682 \\ 324.3406 \end{bmatrix}$$

A matrix T^F whose rows form a minimal generating set of the non-negative left kernel of the matrix product $G^F B$ is given by

$$T^F = \begin{bmatrix} 5 & 1 & 0 & 0 & 0 & 0 \\ 5 & 0 & 0 & 2 & 0 & 0 \\ 5 & 0 & 0 & 0 & 0 & 0.0898 \\ 0 & 1 & 2.8571 & 0 & 0 & 0 \\ 0 & 0 & 2.8571 & 2 & 0 & 0 \\ 0 & 0 & 2.8571 & 0 & 0 & 0.0898 \\ 0 & 1 & 0 & 0 & 0.0898 & 0 \\ 0 & 0 & 0 & 2 & 0.0898 & 0 \\ 0 & 0 & 0 & 0 & 0.0898 & 0.0898 \end{bmatrix}$$

A point $x \in R[G^F, \rho^F]$ can be represented in the form (2) from the vertices

$$x_1 = \begin{bmatrix} 3.8957 \\ 1.1043 \end{bmatrix}, x_2 = \begin{bmatrix} 0.5556 \\ 4.4444 \end{bmatrix}, x_3 = \begin{bmatrix} -0.6055 \\ 1.1271 \end{bmatrix},$$

$$x_4 = \begin{bmatrix} -0.7791 \\ -0.2209 \end{bmatrix}, x_5 = \begin{bmatrix} 0.6667 \\ -1.6667 \end{bmatrix}, x_6 = \begin{bmatrix} 3.7365 \\ -0.1318 \end{bmatrix}$$

partitioning $R[G^F, \rho^F]$ into six regions: $\mathcal{X}_1$ defined by x_1 and x_2, $\mathcal{X}_2$ defined by x_2 and x_3, $\mathcal{X}_3$ defined by x_3 and x_4, $\mathcal{X}_4$ defined by x_4 and x_5, $\mathcal{X}_5$ defined by x_5 and x_6, $\mathcal{X}_6$ defined by x_6 and x_1.

Admissible associated controls are given by

$$v_1 = 3.6249,\ v_2 = -21.3833,\ v_3 = -15.2909,$$
$$v_4 = -0.7250,\ v_5 = 4.5450,\ v_6 = 16.9817$$

A piecewise linear control law (39) can then be used to achieve closed-loop positive invariance of $R[G^F, \rho^F]$, with

$$F_1 = [2.3786, -5.1086],\ F_2 = [13.2210, -6.4639],$$
$$F_3 = [4.1449, -11.3400],\ F_4 = [1.5300\ -2.1150],$$
$$F_5 = [4.5123, -0.9221],\ F_6 = [4.1450, -11.3400]$$

It turns out that positive invariance can be achieved by a linear state feedback as well. Indeed, the conditions of Theorem 5.1 are satisfied with

$$H = \begin{bmatrix} -10.0000 & 0 & 0 & 0 & 0 & 0.0293 \\ 0 & -10.0000 & 0 & 0 & 0.1465 & 0 \\ 0 & 1.4213 & -2.4116 & 0.1584 & 0 & 0.0010 \\ 5.9777 & 0 & 0 & -7.1152 & 0.0098 & 0 \\ 0 & 0 & 0 & 0 & -0.5000 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.5000 \end{bmatrix}, u_c = 0$$

An admissible control law is then $u(t) = Fx(t)$, with $F = [4.1449 - 11.3400]$. The polyhedra $R[G, \rho]$ and $R[G^F, \rho^F]$ are represented in Fig. 2.

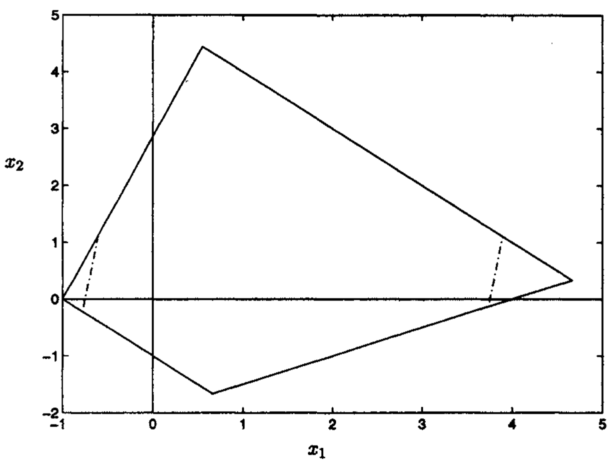

Fig. 2. $R[G, \rho]$ and $R[G^F, \rho^F]$.

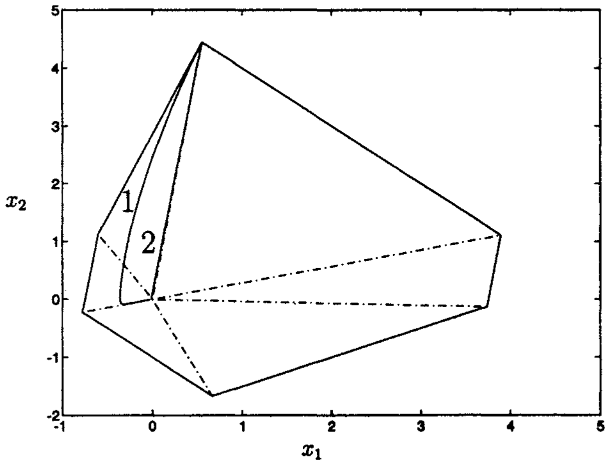

Fig. 3. $R[G^F, \rho^F]$ divided into regions, with two trajectories starting from x_2, resulting from (1) the piecewise linear control law and (2) the linear control law.

$R[G^F, \rho^F]$ divided into regions is represented in Fig. 3, together with two trajectories starting from the vertex x_2.

8. Conclusion

(A,B)-invariance of general convex polyhedral sets with respect to continuous-time linear systems has been characterised by linear matrix relations which only depend on the system matrices and on the considered polyhedral set. Unlike other characterisations found in the literature, such relations can be directly tested on the original system, with no need to rely on discrete-time approximating system. Their satisfaction guarantees the existence of a continuous Lipschitz control law for which the polyhedron is positively invariant. This characterisation has been also extended to treat issues of frequent practical interest, namely constrained controls and additively disturbed systems.

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