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Critical behavior of the compactified $\lambda\phi^4$ theory

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We investigate the critical behavior of the N-component Euclidean $\lambda\phi^4$ model, in the large N limit, in three situations: confined between two parallel planes a distance L apart from one another; confined to an infinitely long cylinder having a square transversal section of area L^2; and to a cubic box of volume L^3. Taking the mass term in the form $m_0^2=\alpha(T-T_0)$, we retrieve Ginzburg–Landau models which are supposed to describe samples of a material undergoing a phase transition, respectively, in the form of a film, a wire and of a grain, whose bulk transition temperature (T_0) is known. We obtain equations for the critical temperature as functions of L and of T_0, and determine the limiting sizes sustaining the transition. © *2005 American Institute of Physics.* [DOI: 10.1063/1.1828589]

I. INTRODUCTION

Models with fields confined in spatial dimensions play important roles both in field theory and in quantum mechanics. Relevant examples are the Casimir effect and superconducting films, where confinement is carried on by appropriate boundary conditions. For Euclidean field theories, imaginary time and the spatial coordinates are treated exactly on the same footing, so that an extended Matsubara formalism can be applied for dealing with the breaking of invariance along any one of the spatial directions.

Relying on this fact, in the present work we discuss the critical behavior of the Euclidean $\lambda\varphi^4$ model compactified in one, two, and three spatial dimensions. We implement the spontaneous symmetry breaking by taking the bare mass coefficient in the Lagrangian parametrized as $m_0^2=\alpha(T-T_0)$, with $\alpha>0$ and the parameter T varying in an interval containing T_0. With this choice, considering the system confined between two parallel planes a distance L apart from one another, in an infinitely long square cylinder with transversal section area $A=L^2$, and in a cube of volume $V=L^3$, in dimension $D=3$, we obtain Ginzburg–Landau models describing phase transitions in samples of a material in the form of a film, a wire and a grain, respectively, T_0 standing for the bulk transition temperature. Such descriptions apply to physical circumstances where no gauge fluctuations need to be considered.

We start recapitulating the general procedure developed in Ref. 1 to treat the massive $(\lambda\varphi^4)_D$ theory in Euclidean space, compactified in a d-dimensional subspace, with $d\leqslant D$. This permits to extend to an arbitrary subspace some results in the literature for finite temperature field theory[2] and for the behavior of field theories in the presence of spatial boundaries.[3,4] We shall consider the vector N-component $(\lambda\varphi^4)_D$ Euclidean theory at leading order in $1/N$, thus allowing for nonper-

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turbative results, the system being submitted to the constraint of compactification of a d-dimensional subspace. After describing the general formalism, we readdress the renormalization procedure we use treating the simpler situation of $d=1$, which corresponds to the system confined between parallel planes (a film), analyzed in Ref. 5 for the case of two components, $N=2$. We then focus on two other particularly interesting cases of $d=2$ and $d=3$, in the three-dimensional Euclidean space, corresponding, respectively, to the system confined to an infinitely long cylinder with square transversal section (a wire) and to a finite cubic box (a grain). Extending the investigation to these new cases demands further developments in the subject of multidimensional Epstein functions.

For these situations, in the framework of the Ginzburg–Landau model we derive equations for the critical temperature as a function of the confining dimensions. For a film, we show that the critical temperature decreases linearly with the inverse of the film thickness while, for a square wire and for a cubic grain, we obtain that the critical temperatures decrease linearly with the inverse of the side of the square and with the inverse of the edge of the cube, respectively, but with larger coefficients. In all cases, we are able to calculate the minimal system size (thickness, transversal section area, or volume) below which the phase transition does not take place.

II. THE COMPACTFIED MODEL

In this section we review the analytical methods of compactification of the N-component Euclidean $\lambda\varphi^4$ model developed in Ref. 1 We consider the model described by the Hamiltonian density,

$$\mathcal{H} = \frac{1}{2}\partial_\mu\varphi_a\partial^\mu\varphi_a + \frac{1}{2}\overline{m}_0^2\varphi_a\varphi_a + \frac{\lambda}{N}(\varphi_a\varphi_a)^2, \tag{1}$$

in Euclidean D-dimensional space, confined to a d-dimensional spatial rectangular box of sides L_j, $j=1,2,\ldots,d$. In the above equation λ is the *renormalized* coupling constant, $\overline{m}_0^2$ is a boundary-modified mass parameter depending on $\{L_i\}$ $i=1,2,\ldots,d$, in such a way that

$$\lim_{\{L_i\}\to\infty} \overline{m}_0^2(L_1,\ldots,L_d) = m_0^2(T) \equiv \alpha(T-T_0), \tag{2}$$

$m_0^2(T)$ being the constant mass parameter present in the usual free-space Ginzburg–Landau model. In Eq. (2), T_0 represents the bulk transition temperature. Summation over repeated "color" indices a is assumed. To simplify the notation in the following we drop out the color indices, summation over them being understood in field products. We will work in the approximation of neglecting boundary corrections to the coupling constant. A precise definition of the boundary-modified mass parameter will be given later for the situation of $D=3$ with $d=1$, $d=2$, and $d=3$, corresponding, respectively, to a film of thickness L_1, to a wire of rectangular section $L_1\times L_2$ and to a grain of volume $L_1\times L_2\times L_3$.

We use Cartesian coordinates $\mathbf{r}=(x_1,\ldots,x_d,\mathbf{z})$, where $\mathbf{z}$ is a $(D-d)$-dimensional vector, with corresponding momentum $\mathbf{k}=(k_1,\ldots,k_d,\mathbf{q})$, $\mathbf{q}$ being a $(D-d)$-dimensional vector in momentum space. Then the generating functional of correlation functions has the form

$$\mathcal{Z} = \int \mathcal{D}\varphi^\dagger\,\mathcal{D}\varphi\,\exp\left(-\int_0^{\mathbf{L}} \mathrm{d}^d r \int \mathrm{d}^{D-d}\mathbf{z}\,\mathcal{H}(\varphi,\nabla\varphi)\right), \tag{3}$$

where $\mathbf{L}=(L_1,\ldots,L_d)$, and we are allowed to introduce a generalized Matsubara prescription, performing the following multiple replacements (compactification of a d-dimensional subspace):

$$\int \frac{\mathrm{d}k_i}{2\pi} \to \frac{1}{L_i}\sum_{n_i=-\infty}^{+\infty}, \quad k_i \to \frac{2n_i\pi}{L_i}, \quad i=1,2,\ldots,d. \tag{4}$$

A simpler situation is the system confined simultaneously between two parallel planes a distance L_1 apart from one another normal to the x_1 axis and two other parallel planes, normal to the x_2 axis separated by a distance L_2 (a "wire" of rectangular section).

We start from the well-known expression for the one-loop contribution to the zero-temperature effective potential,[6]

$$U_1(\varphi_0) = \sum_{s=1}^{\infty} \frac{(-1)^{s+1}}{2s}[12\lambda\varphi_0^2]^s \int \frac{\mathrm{d}^D k}{(2\pi)^D}\frac{1}{(k^2+m^2)^s}, \tag{5}$$

where m is the physical mass and φ_0 is the normalized vacuum expectation value of the field (the classical field). In the following, to deal with dimensionless quantities in the regularization procedures, we introduce parameters

$$c = \frac{m}{2\pi\mu}, \quad b_i = \frac{1}{L_i\mu}, \quad g = \frac{\lambda}{4\pi^2\mu^{4-D}}, \quad \phi_0^2 = \frac{\varphi_0^2}{\mu^{D-2}}, \tag{6}$$

where μ is a mass scale. In terms of these parameters and performing the replacements (4), the one-loop contribution to the effective potential can be written in the form

$$U_1(\phi_0,b_1,\ldots,b_d) = \mu^D b_1\cdots b_d \sum_{s=1}^{\infty}\frac{(-1)^s}{2s}[12g\phi_0^2]^s \sum_{n_1,\ldots,n_d=-\infty}^{+\infty}\int \frac{\mathrm{d}^{D-d}q'}{(b_1^2n_1^2+\cdots+b_d^2n_d^2+c^2+\mathbf{q}'^2)^s}, \tag{7}$$

where $\mathbf{q}'=\mathbf{q}/2\pi\mu$ is dimensionless. Using a well-known dimensional regularization formula[7] to perform the integration over the $(D-d)$ noncompactified momentum variables, we obtain

$$U_1(\phi_0,b_1,\ldots,b_d) = \mu^D b_1\cdots b_d \sum_{s=1}^{\infty} f(D,d,s)[12g\phi_0^2]^s A_d^{c^2}\left(s-\frac{D-d}{2};b_1,\ldots,b_d\right), \tag{8}$$

where

$$f(D,d,s) = \pi^{(D-d)/2}\frac{(-1)^{s+1}}{2s\Gamma(s)}\Gamma\left(s-\frac{D-d}{2}\right) \tag{9}$$

and

$$\begin{aligned} A_d^{c^2}(\nu;b_1,\ldots,b_d) &= \sum_{n_1,\ldots,n_d=-\infty}^{+\infty}(b_1^2n_1^2+\cdots+b_d^2n_d^2+c^2)^{-\nu} \\ &= \frac{1}{c^{2\nu}} + 2\sum_{i=1}^{d}\sum_{n_i=1}^{\infty}(b_i^2n_i^2+c^2)^{-\nu} \\ &\quad + 2^2\sum_{i<j=1}^{d}\sum_{n_i,n_j=1}^{\infty}(b_i^2n_i^2+b_j^2n_j^2+c^2)^{-\nu}+\cdots \\ &\quad + 2^d\sum_{n_1,\ldots,n_d=1}^{\infty}(b_1^2n_1^2+\cdots+b_d^2n_d^2+c^2)^{-\nu}. \end{aligned} \tag{10}$$

Next we can proceed generalizing to several dimensions the mode-sum regularization prescription described in Ref. 8. This generalization has been done in Ref. 1 and we briefly describe here its principal steps. From the identity

$$\frac{1}{\Delta^{\nu}} = \frac{1}{\Gamma(\nu)}\int_0^{\infty} \mathrm{d}t\, t^{\nu-1}e^{-\Delta t}, \tag{11}$$

and using the following representation for Bessel functions of the third kind, K_{ν},

$$2(a/b)^{\nu/2}K_{\nu}(2\sqrt{ab}) = \int_0^{\infty} \mathrm{d}x\, x^{\nu-1}e^{-(a/x)-bx}, \tag{12}$$

we obtain after some rather long but straightforward manipulations,[1]

$$\begin{aligned} A_d^{c^2}(\nu;b_1,\ldots,b_d) = \frac{2^{\nu-(d/2)+1}\pi^{2\nu-(d/2)}}{b_1\cdots b_d\Gamma(\nu)}\Bigg[& 2^{\nu-(d/2)-1}\Gamma\left(\nu-\frac{d}{2}\right)(2\pi c)^{d-2\nu} \\ & + 2\sum_{i=1}^{d}\sum_{n_i=1}^{\infty}\left(\frac{n_i}{2\pi c b_i}\right)^{\nu-(d/2)}K_{\nu-(d/2)}\left(\frac{2\pi c n_i}{b_i}\right) + \cdots \\ & + 2^d\sum_{n_1,\ldots,n_d=1}^{\infty}\left(\frac{1}{2\pi c}\sqrt{\frac{n_1^2}{b_1^2}+\cdots+\frac{n_d^2}{b_d^2}}\right)^{\nu-(d/2)}K_{\nu-(d/2)}\left(2\pi c\sqrt{\frac{n_1^2}{b_1^2}+\cdots+\frac{n_d^2}{b_d^2}}\right)\Bigg]. \end{aligned} \tag{13}$$

Taking $\nu = s-(D-d)/2$ in Eq. (13) and inserting it in Eq. (8), we obtain the one-loop correction to the effective potential in D dimensions with a compactified d-dimensional subspace in the form (recovering the dimensionful parameters)

$$\begin{aligned} U_1(\varphi_0,L_1,\ldots,L_d) = \sum_{s=1}^{\infty}[12g\phi_0^2]^s h(D,s)\Bigg[& 2^{s-(D/2)-2}\Gamma\left(s-\frac{D}{2}\right)m^{D-2s} \\ & + \sum_{i=1}^{d}\sum_{n_i=1}^{\infty}\left(\frac{m}{L_i n_i}\right)^{(D/2)-s}K_{(D/2)-s}(mL_i n_i) \\ & + 2\sum_{i<j=1}^{d}\sum_{n_i,n_j=1}^{\infty}\left(\frac{m}{\sqrt{L_i^2n_i^2+L_j^2n_j^2}}\right)^{(D/2)-s}K_{(D/2)-s}\left(m\sqrt{L_i^2n_i^2+L_j^2n_j^2}\right) + \cdots \\ & + 2^{d-1}\sum_{n_1,\ldots,n_d=1}^{\infty}\left(\frac{m}{\sqrt{L_1^2n_1^2+\cdots+L_d^2n_d^2}}\right)^{(D/2)-s}K_{(D/2)-s}\left(m\sqrt{L_1^2n_1^2+\cdots+L_d^2n_d^2}\right)\Bigg], \end{aligned} \tag{14}$$

with

$$h(D,s) = \frac{1}{2^{D/2+s-1}\pi^{D/2}}\frac{(-1)^{s+1}}{s\Gamma(s)}. \tag{15}$$

Criticality is attained when the inverse squared correlation length, $\xi^{-2}(L_1,\ldots,L_d,\varphi_0)$, vanishes in the large-$N$ gap equation,

$$\xi^{-2}(L_1,\ldots,L_d,\varphi_0) = \overline{m}_0^2 + 12\lambda\varphi_0^2 + \frac{24\lambda}{L_1\cdots L_d}\sum_{n_1,\ldots,n_d=-\infty}^{\infty}\int\frac{\mathrm{d}^{D-d}q}{(2\pi)^{D-d}} \times \frac{1}{\mathbf{q}^2 + \left(\frac{2\pi n_1}{L_1}\right)^2 + \cdots + \left(\frac{2\pi n_d}{L_d}\right)^2 + \xi^{-2}(L_1,\ldots,L_d,\varphi_0)}, \tag{16}$$

where φ_0 is the normalized vacuum expectation value of the field (different from zero in the ordered phase). In the disordered phase, φ_0 vanishes and the inverse correlation length equals the physical mass, given below by Eq. (18). Recalling the condition,

$$\left.\frac{\partial^2}{\partial\varphi_0^2}U(D,L_1,L_2)\right|_{\varphi_0=0} = m^2, \tag{17}$$

where U is the sum of the tree-level and one-loop contributions to the effective potential (remembering that at the large-N limit it is enough to take the one-loop contribution to the mass), we obtain

$$\begin{aligned} m^2(L_1,\ldots,L_d) = \overline{m}_0^2(L_1,\ldots,L_d) + \frac{24\lambda}{(2\pi)^{D/2}}\Bigg[&\sum_{i=1}^{d}\sum_{n_i=1}^{\infty}\left(\frac{m}{L_i n_i}\right)^{(D/2)-1}K_{(D/2)-1}(mL_in_i) \\ &+ 2\sum_{i<j=1}^{d}\sum_{n_i,n_j=1}^{\infty}\left(\frac{m}{\sqrt{L_i^2n_i^2+L_j^2n_j^2}}\right)^{(D/2)-1}K_{(D/2)-1}\left(m\sqrt{L_i^2n_i^2+L_j^2n_j^2}\right) + \cdots \\ &+ 2^{d-1}\sum_{n_1,\ldots,n_d=1}^{\infty}\left(\frac{m}{\sqrt{L_1^2n_1^2+\cdots+L_d^2n_d^2}}\right)^{(D/2)-1}K_{(D/2)-1}\left(m\sqrt{L_1^2n_1^2+\cdots+L_d^2n_d^2}\right)\Bigg]. \end{aligned} \tag{18}$$

Notice that, in writing Eq. (18), we have suppressed the parcel $2^{-(D/2)-1}\Gamma[1-(D/2)]m^{D-2}$ from its square brackets, the parcel that emerges from the first term in the square bracket of Eq. (14). This expression, which does not depend explicitly on L_i, diverges for D even due to the poles of the gamma function; in this case, this parcel is subtracted to get a renormalized mass equation. For D odd, $\Gamma[1-(D/2)]$ is finite but we also subtract this term (corresponding to a finite renormalization) for sake of uniformity; besides, for $D\geqslant 3$, the factor m^{D-2} does not contribute in the criticality.

The vanishing of Eq. (18) defines criticality for our compactified system. We claim that, taking $d=1$, $d=2$, and $d=3$ with $D=3$, we are able to describe, respectively, the critical behavior of samples of materials in the form of films, wires, and grains. Notice that the parameter m on the right-hand side of Eq. (18) is the boundary-modified mass $m(L_1,\ldots,L_d)$, which means that Eq. (18) is a self-consistency equation, a very complicated modified Schwinger–Dyson equation for the mass, not soluble by algebraic means. Nevertheless, as we will see in the next sections, a solution is possible at criticality, which allows us to obtain a closed formula for the boundary-dependent critical temperature.

III. CRITICAL BEHAVIOR FOR FILMS

We now consider the simplest particular case of the compactification of only one spatial dimension, with the system confined between two parallel planes a distance L apart from one another. This case, which has already been considered in Ref. 5 concerning with the two-component model, is reanalyzed here to set the required renormalization procedure in the proper large-N grounds and, also, for the sake of completeness. Thus, from Eq. (18), taking $d=1$, we get in the disordered phase

$$m^2(L) = \overline{m}_0^2(L) + \frac{24\lambda}{(2\pi)^{D/2}} \sum_{n=1}^{\infty} \left(\frac{m}{nL}\right)^{(D/2)-1} K_{(D/2)-1}(nLm), \tag{19}$$

where L $(=L_1)$ is the separation between the planes, the film thickness. If we limit ourselves to the neighborhood of criticality $(m^2 \approx 0)$ and consider L finite and sufficiently small, we may use an asymptotic formula for small values of the argument of Bessel functions,

$$K_\nu(z) \approx \frac{1}{2}\Gamma(|\nu|)\left(\frac{z}{2}\right)^{-|\nu|} \quad (z \approx 0), \tag{20}$$

and Eq. (19) reduces, for $D > 3$, to

$$m^2(L) \approx \overline{m}_0^2(L) + \frac{6\lambda}{\pi^{D/2} L^{D-2}} \Gamma\left(\frac{D}{2} - 1\right) \zeta(D-2), \tag{21}$$

where $\zeta(D-2)$ is the Riemann *zeta*-function, defined for $\mathrm{Re}\{D-2\} > 1$ by the series

$$\zeta(D-2) = \sum_{n=1}^{\infty} \frac{1}{n^{D-2}}. \tag{22}$$

It is worth mentioning that for $D=4$, taking $m^2(L)=0$ and making the appropriate changes ($L \to \beta, \lambda \to \lambda/4!$), Eq. (21) is *formally identical* to the high-temperature (low values of β) critical equation obtained in Ref. 9, thus providing a check of our calculations.

For $D=3$, Eq. (21) can be made physically meaningful by a regularization procedure as follows. We consider the analytic continuation of the *zeta*-function, leading to a meromorphic function having only one simple pole at $z=1$, which satisfies the reflection formula

$$\zeta(z) = \frac{1}{\Gamma(z/2)} \Gamma\left(\frac{1-z}{2}\right) \pi^{z-1/2} \zeta(1-z). \tag{23}$$

Next, remembering the formula

$$\lim_{z \to 1} \left[\zeta(z) - \frac{1}{z-1} \right] = \gamma, \tag{24}$$

where $\gamma \approx 0.5772$ is the Euler–Mascheroni constant, we define the L-dependent bare mass for $D \approx 3$, in such a way that the pole at $D=3$ in Eq. (21) is suppressed, that is we take

$$\overline{m}_0^2(L) \approx M - \frac{1}{(D-3)} \frac{6\lambda}{\pi L}, \tag{25}$$

where M is independent of D. To fix the finite term, we make the simplest choice satisfying (2),

$$M = m_0^2(T) = \alpha(T - T_0), \tag{26}$$

T_0 being the bulk critical temperature. In this case, using Eq. (25) in Eq. (21) and taking the limit as $D \to 3$, the L-dependent renormalized mass term in the vicinity of criticality becomes

$$m^2(L) \approx \alpha(T - T_c(L)), \tag{27}$$

where the modified, L-dependent, transition temperature is given by

$$T_c(L) = T_0 - C_1 \frac{\lambda}{\alpha L}, \tag{28}$$

L being the thickness of the film, with the constant C_1 given by

$$C_1 = \frac{6\gamma}{\pi} \approx 1.1024. \tag{29}$$

From this equation, we see that for L smaller than

$$L_{\min} = C_1 \frac{\lambda}{\alpha T_0}, \tag{30}$$

$T_c(L)$ becomes negative, meaning that the transition does not occur.[5]

IV. CRITICAL BEHAVIOR FOR WIRES

We now focus on the situation where two spatial dimensions are compactified. From Eq. (18), taking $d=2$, we get (in the disordered phase)

$$\begin{aligned} m^2(L_1,L_2) = \overline{m}_0^2(L_1,L_2) + \frac{24\lambda}{(2\pi)^{D/2}} \Bigg[& \sum_{n=1}^{\infty} \left(\frac{m}{nL_1}\right)^{(D/2)-1} K_{(D/2)-1}(nL_1 m) \\ & + \sum_{n=1}^{\infty} \left(\frac{m}{nL_2}\right)^{(D/2)-1} K_{(D/2)-1}(nL_2 m) \\ & + 2\sum_{n_1,n_2=1}^{\infty} \left(\frac{m}{\sqrt{L_1^2 n_1^2 + L_2^2 n_2^2}}\right)^{(D/2)-1} K_{(D/2)-1}\left(m\sqrt{L_1^2 n_1^2 + L_2^2 n_2^2}\right) \Bigg]. \end{aligned} \tag{31}$$

If we limit ourselves to the neighborhood of criticality, $m^2 \approx 0$, and taking both L_1 and L_2 finite and sufficiently small, we may use Eq. (20) to rewrite Eq. (31) as

$$m^2(L_1,L_2) \approx \overline{m}_0^2(L_1,L_2) + \frac{6\lambda}{\pi^{D/2}} \Gamma\left(\frac{D}{2} - 1\right) \left[\left(\frac{1}{L_1^{D-2}} + \frac{1}{L_2^{D-2}} \right) \zeta(D-2) + 2E_2\left(\frac{D-2}{2}; L_1, L_2\right) \right], \tag{32}$$

where $E_2[(D-2)/2; L_1, L_2]$ is the generalized (multidimensional) Epstein *zeta*-function defined by

$$E_2\left(\frac{D-2}{2}; L_1, L_2\right) = \sum_{n_1,n_2=1}^{\infty} [L_1^2 n_1^2 + L_2^2 n_2^2]^{-[(D-2)/2]}, \tag{33}$$

for $\mathrm{Re}\{D\} > 3$.

As mentioned before, the Riemann *zeta*-function $\zeta(D-2)$ has an analytical extension to the whole complex D-plane, having a unique simple pole (of residue 1) at $D=3$. One can also construct analytical continuations (and recurrence relations) for the multidimensional Epstein functions which permit to write them in terms of Kelvin and Riemann *zeta*-functions. To start one considers the analytical continuation of the Epstein–Hurwitz *zeta*-function given by[8]

$$\sum_{n=1}^{\infty} (n^2 + p^2)^{-\nu} = -\frac{1}{2} p^{-2\nu} + \frac{\sqrt{\pi}}{2p^{2\nu-1}\Gamma(\nu)} \left[\Gamma\left(\nu - \frac{1}{2}\right) + 4\sum_{n=1}^{\infty} (\pi p n)^{\nu-1/2} K_{\nu-1/2}(2\pi p n) \right]. \tag{34}$$

Using this relation to perform one of the sums in (33) leads immediately to the question of which sum is first evaluated. As it is done in Ref. 10, whatever the sum one chooses to perform first, the manifest $L_1 \leftrightarrow L_2$ symmetry of Eq. (33) is lost; to overcome such an obstacle, in order to preserve this symmetry, we adopt here a symmetrized summation generalizing the prescription introduced in Ref. 1 for the case of many variables.

To derive an analytical continuation and symmetrized recurrence relations for the multidimensional Epstein functions, we start by taking these functions defined as the symmetrized summations

$$E_d(\nu;L_1,\ldots,L_d)=\frac{1}{d!}\sum_{\sigma}\sum_{n_1=1}^{\infty}\cdots\sum_{n_d=1}^{\infty}[\sigma_1^2n_1^2+\cdots+\sigma_d^2n_d^2]^{-\nu}, \tag{35}$$

where $\sigma_i=\sigma(L_i)$, with σ running in the set of all permutations of the parameters $L_1,\ldots,L_d$, and the summations over $n_1,\ldots,n_d$ being taken in the given order. Applying (34) to perform the sum over n_d, one gets

$$E_d(\nu;L_1,\ldots,L_d)=-\frac{1}{2d}\sum_{i=1}^{d}E_{d-1}(\nu;\ldots,\widehat{L_i},\ldots)+\frac{\sqrt{\pi}}{2\,d\Gamma(\nu)}\Gamma\left(\nu-\frac{1}{2}\right)\sum_{i=1}^{d}\frac{1}{L_i}E_{d-1}\left(\nu-\frac{1}{2};\ldots,\widehat{L_i},\ldots\right)$$
$$+\frac{2\sqrt{\pi}}{d\Gamma(\nu)}W_d\left(\nu-\frac{1}{2},L_1,\ldots,L_d\right), \tag{36}$$

where the hat over the parameter L_i in the functions E_{d-1} means that it is excluded from the set $\{L_1,\ldots,L_d\}$ (the others being the $d-1$ parameters of E_{d-1}), and

$$W_d(\eta;L_1,\ldots,L_d)=\sum_{i=1}^{d}\frac{1}{L_i}\sum_{n_1,\ldots,n_d=1}^{\infty}\left(\frac{\pi n_i}{L_i\sqrt{(\cdots+\widehat{L_i^2n_i^2}+\cdots)}}\right)^{\eta}K_\eta\left(\frac{2\pi n_i}{L_i}\sqrt{(\cdots+\widehat{L_i^2n_i^2}+\cdots)}\right), \tag{37}$$

with $(\cdots+\widehat{L_i^2n_i^2}+\cdots)$ representing the sum $\Sigma_{j=1}^{d}L_j^2n_j^2-L_i^2n_i^2$. In particular, noticing that $E_1(\nu;L_j)=L_j^{-2\nu}\zeta(2\nu)$, one finds

$$E_2\left(\frac{D-2}{2};L_1^2,L_2^2\right)=-\frac{1}{4}\left(\frac{1}{L_1^{D-2}}+\frac{1}{L_2^{D-2}}\right)\zeta(D-2)+\frac{\sqrt{\pi}\Gamma\left(\frac{D-3}{2}\right)}{4\Gamma\left(\frac{D-2}{2}\right)}\left(\frac{1}{L_1L_2^{D-3}}+\frac{1}{L_1^{D-3}L_2}\right)\zeta(D-3)$$
$$+\frac{\sqrt{\pi}}{\Gamma\left(\frac{D-2}{2}\right)}W_2\left(\frac{D-3}{2};L_1,L_2\right), \tag{38}$$

which is a meromorphic function of D, symmetric in the parameters L_1 and L_2 as Eq. (33) suggests.

Using the above expression, Eq. (32) can be rewritten as

$$m^2(L_1,L_2)\approx\overline{m}_0^2(L_1,L_2)+\frac{3\lambda}{\pi^{D/2}}\left[\left(\frac{1}{L_1^{D-2}}+\frac{1}{L_2^{D-2}}\right)\Gamma\left(\frac{D-2}{2}\right)\zeta(D-2)\right.$$
$$\left.+\sqrt{\pi}\left(\frac{1}{L_1L_2^{D-3}}+\frac{1}{L_1^{D-3}L_2}\right)\Gamma\left(\frac{D-3}{2}\right)\zeta(D-3)+2\sqrt{\pi}W_2\left(\frac{D-3}{2};L_1,L_2\right)\right]. \tag{39}$$

This equation presents no problems for $3<D<4$ but, for $D=3$, the first and second terms between the square brackets of Eq. (39) are divergent due to the ζ-function and Γ-function, respectively. We can deal with divergences remembering the property in Eq. (24) and using the expansion of $\Gamma[(D-3)/2]$ around $D=3$,

$$\Gamma\left(\frac{D-3}{2}\right)\approx\frac{2}{D-3}+\Gamma'(1), \tag{40}$$

$\Gamma'(z)$ standing for the derivative of the Γ-function with respect to z. For $z=1$ it coincides with the Euler digamma-function $\psi(1)$, which has the particular value $\psi(1)=-\gamma$. We notice however, that differently from the case treated in the preceding section, where a renormalization procedure was needed, here the two divergent terms generated by the use of formulas (24) and (40) cancel exactly between them. No renormalization is needed. Thus, for $D=3$, taking the bare mass given by $\overline{m}_0^2(L_1,L_2)=\alpha(T-T_0)$, we obtain the renormalized boundary-dependent mass term in the form

$$m^2(L_1,L_2) \approx \alpha(T - T_c(L_1,L_2)), \tag{41}$$

with the boundary-dependent critical temperature given by

$$T_c(L_1,L_2) = T_0 - \frac{9\lambda\gamma}{2\pi\alpha}\left(\frac{1}{L_1} + \frac{1}{L_2}\right) - \frac{6\lambda}{\pi\alpha}W_2(0;L_1,L_2), \tag{42}$$

where

$$W_2(0;L_1,L_2) = \sum_{n_1,n_2=1}^{\infty}\left\{\frac{1}{L_1}K_0\left(2\pi\frac{L_2}{L_1}n_1n_2\right) + \frac{1}{L_2}K_0\left(2\pi\frac{L_1}{L_2}n_1n_2\right)\right\}. \tag{43}$$

The quantity $W_2(0;L_1,L_2)$, appearing in Eq. (42), involves complicated double sums, very difficult to handle for $L_1 \neq L_2$; in particular, it is not possible to take limits such as $L_i \to \infty$. For this reason we will restrict ourselves to the case $L_1=L_2$. For a wire with the square transversal section, we have $L_1=L_2=L=\sqrt{A}$ and Eq. (42) reduces to

$$T_c(A) = T_0 - C_2\frac{\lambda}{\alpha\sqrt{A}}, \tag{44}$$

where C_2 is a constant given by

$$C_2 = \frac{9\gamma}{\pi} + \frac{12}{\pi}\sum_{n_1,n_2=1}^{\infty}K_0(2\pi n_1n_2) \approx 1.6571. \tag{45}$$

We see that the critical temperature of the square wire depends on the bulk critical temperature and the Ginzburg–Landau parameters α and λ (which are characteristics of the material constituting the wire), and also on the area of its cross section. Since T_c decreases linearly with the inverse of the side of the square, this suggests that there is a minimal area for which $T_c(A_{\min})=0$,

$$A_{\min} = \left(C_2\frac{\lambda}{\alpha T_0}\right)^2; \tag{46}$$

for square wires of the transversal section areas smaller than this value, in the context of our model the transition should be suppressed. On topological grounds, we expect that (apart from appropriate coefficients) our result should be independent of the transverse section shape of the wire, at least for transversal sectional regular polygons.

V. CRITICAL BEHAVIOR FOR GRAINS

We now turn our attention to the case where all three spatial dimensions are compactified, corresponding to the system confined in a box of sides L_1,L_2,L_3. Taking $d=3$ in Eq. (18) and using Eq. (20), we obtain (for sufficiently small L_1,L_2,L_3 and in the neighborhood of classicality, $m^2 \approx 0$)

$$m^2(L_1,L_2,L_3) \approx \overline{m}_0^2(L_1,L_2,L_3) + \frac{6\lambda}{\pi^{D/2}}\Gamma\left(\frac{D-2}{2}\right)\left[\sum_{i=1}^{3}\frac{\zeta(D-2)}{L_i^{D-2}} + 2\sum_{i<j=1}^{3}E_2\left(\frac{D-2}{2};L_i,L_j\right) + 4E_3\left(\frac{D-2}{2};L_1,L_2,L_3\right)\right], \tag{47}$$

where $E_3(\nu;L_1,L_2,L_3)=\sum_{n_1,n_2,n_3=1}^{\infty}[L_1^2n_1^2+L_2^2n_2^2+L_3^2n_3^2]^{-\nu}$ and the functions E_2 are given by Eq. (38).

The analytical structure of the function $E_3[(D-2)/2;L_1,L_2,L_3]$ can be obtained from the general symmetrized recurrence relation given by Eqs. (36) and (37); explicitly, one has

$$E_3\left(\frac{D-2}{2};L_1,L_2,L_3\right) = -\frac{1}{6}\sum_{i<j=1}^{3}E_2\left(\frac{D-2}{2};L_i,L_j\right) + \frac{\sqrt{\pi}\Gamma\left(\frac{D-3}{2}\right)}{6\Gamma\left(\frac{D-2}{2}\right)}\sum_{i,j,k=1}^{3}\frac{(1+\varepsilon_{ijk})}{2}\frac{1}{L_i} \times E_2\left(\frac{D-2}{2};L_j,L_k\right) + \frac{2\sqrt{\pi}}{3\Gamma\left(\frac{D-2}{2}\right)}W_3\left(\frac{D-3}{2};L_1,L_2,L_3\right), \tag{48}$$

where ε_{ijk} is the totally antisymmetric symbol and the function W_3 is a particular case of Eq. (37). Using Eqs. (38) and (48), the boundary dependent mass can be written as

$$\begin{aligned} m^2(L_1,L_2,L_3) \approx \overline{m}_0^2(L_1,L_2,L_3) + \frac{6\lambda}{\pi^{D/2}}\Bigg[&\frac{1}{3}\Gamma\left(\frac{D-2}{2}\right)\sum_{i=1}^{3}\frac{1}{L_i^{D-2}}\zeta(D-2) + \frac{\sqrt{\pi}}{6}\zeta(D-3) \\ &\times\sum_{i<j=1}^{3}\left(\frac{1}{L_i^{D-3}L_j}+\frac{1}{L_j^{D-3}L_i}\right)\Gamma\left(\frac{D-3}{2}\right) + \frac{4\sqrt{\pi}}{3}\sum_{i<j=1}^{3}W_2\left(\frac{D-3}{2};L_i,L_j\right) \\ &+\frac{\pi}{6}\zeta(D-4)\Gamma\left(\frac{D-4}{2}\right)\sum_{i,j,k=1}^{3}\frac{(1+\varepsilon_{ijk})}{2}\frac{1}{L_i}\left(\frac{1}{L_j^{D-4}L_k}+\frac{1}{L_k^{D-4}L_j}\right) \\ &+\frac{2\pi}{3}\sum_{i,j,k=1}^{3}\frac{(1+\varepsilon_{ijk})}{2}\frac{1}{L_i}W_2\left(\frac{D-4}{2};L_j,L_k\right) + \frac{8\sqrt{\pi}}{3}W_3\left(\frac{D-3}{2};L_1,L_2,L_3\right)\Bigg]. \end{aligned} \tag{49}$$

The first two terms in the square brackets of Eq. (49) diverge as $D\to 3$ due to the poles of the Γ and ζ-functions. However, as it happens in the case of wires, using Eqs. (24) and (40) it can be shown that these divergences cancel exactly one another. After some simplifications, for $D=3$, the boundary dependent mass (49) becomes

$$m^2(L_1,L_2,L_3) \approx \overline{m}_0^2(L_1,L_2,L_3) + \frac{6\lambda}{\pi}\left[\frac{\gamma}{2}\sum_{i=1}^{3}\frac{1}{L_i} + \frac{4}{3}\sum_{i<j=1}^{3}W_2(0;L_i,L_j) + \frac{\pi}{18}\sum_{i,j,k=1}^{3}\frac{(1+\varepsilon_{ijk})}{2}\frac{L_i}{L_jL_k} + \frac{2\sqrt{\pi}}{3}\sum_{i,j,k=1}^{3}\frac{(1+\varepsilon_{ijk})}{2}\frac{1}{L_i}W_2\left(-\frac{1}{2};L_j,L_k\right) + \frac{8}{3}W_3(0;L_1,L_2,L_3)\right]. \tag{50}$$

As before, since no divergences need to be suppressed, we can take the bare mass given by $\overline{m}_0^2(L_1,L_2,L_3)=\alpha(T-T_0)$ and rewrite the renormalized mass as $m^2(L_1,L_2,L_3)\approx\alpha(T-T_c(L_1,L_2,L_3))$. The expression of $T_c(L_1,L_2,L_3)$ can be easily obtained from Eq. (50), but it is a very complicated formula, involving multiple sums, which makes almost impossible a general analytical study for arbitrary parameters L_1, L_2, L_3; thus, we restrict ourselves to the situation where $L_1 = L_2 = L_3 = L$, corresponding to a cubic box of volume $V = L^3$. In this case, the boundary dependent critical temperature reduces to

$$T_c(V) = T_0 - C_3 \frac{\lambda}{\alpha V^{1/3}}, \tag{51}$$

where the constant C_3 is given by [using that $K_{-1/2}(z) = \sqrt{\pi/2z}e^{-z}$]

$$C_3 = 1 + \frac{9\gamma}{\pi} + \frac{12}{\pi}\sum_{n_1,n_2=1}^{\infty}\frac{e^{-2\pi n_1 n_2}}{n_1} + \frac{48}{\pi}\sum_{n_1,n_2=1}^{\infty} K_0(2\pi n_1 n_2) + \frac{48}{\pi}\sum_{n_1,n_2,n_3=1}^{\infty} K_0\left(2\pi n_1\sqrt{n_2^2+n_3^2}\right) \approx 2.7657. \tag{52}$$

One sees that the minimal volume of the cubic grain sustaining the transition is

$$V_{\min} = \left(C_3 \frac{\lambda}{\alpha T_0}\right)^3. \tag{53}$$

VI. CONCLUSIONS

In this paper we have discussed the spontaneous symmetry breaking of the $(\lambda\phi^4)_D$ theory compactified in $d \leqslant D$ Euclidean dimensions, extending some results of Ref. 1. We have parametrized the bare mass term in the form $m_0^2(T - T_0)$, thus placing the analysis within the Ginzburg–Landau framework. We focused on the situations with $D = 3$ and $d = 1, 2, 3$, corresponding (in the context of condensed matter systems) to films, wires, and grains, respectively, undergoing phase transitions which may be described by (mean-field) Ginzburg–Landau models. This generalizes to more compactified dimensions of previous investigations on the superconducting transition in films, both without[5] and in the presence of a magnetic field.[11] In all cases studied here, in the absence of gauge fluctuations, we found that the boundary-dependent critical temperature decreases linearly with the inverse of the linear dimension L: $T_c(L) = T_0 - C_d \lambda / \alpha L$, where α and λ are the Ginzgurg–Landau parameters, T_0 is the bulk transition temperature, and C_d is a constant equal to 1.1024, 1.6571, and 2.6757 for $d = 1$ (film), $d = 2$ (square wire), and $d = 3$ (cubic grain), respectively. Such behavior suggests the existence of a minimal size of the system below which the transition is suppressed. It is worth mentioning that having the transition temperature scaling with the inverse of the relevant length L for all the cases analyzed (films, wires, and grains) is in accordance with what one learns from finite-size scaling arguments.[12]

These findings seems to be in *qualitative* agreement with results for the existence of a minimal thickness for disappearance of superconductivity in films.[13–16] Experimental investigations in nanowires searching to establish whether there is a limit to how thin a superconducting wire can be, while retaining its superconducting character, have also drawn the attention of researchers; for example, in Ref. 17 the behavior of nanowires has been studied. Similar questions have also been raised concerning the behavior of superconducting nanograins.[18,19] Nevertheless, an important point to be emphasized is that our results are obtained in a field-theoretical framework and do not depend on microscopic details of the material involved nor account for the influence of manufacturing aspects of the sample; in other words, our results emerge solely as a topological effect of the compactification of the Ginzburg–Landau model in a subspace. Detailed microscopic analysis is required if one attempts to account quantitatively for experimental observations which might deviate from our mean field results.

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