New results for deformed defects

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We extend a deformation prescription recently introduced and present some new soluble nonlinear problems for kinks and lumps. In particular, we show how to generate models that present the basic ingredients needed to give rise to dimension bubbles. Also, we show how to deform models that possess lumplike solutions to get to new models that support kinklike solutions.

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I. INTRODUCTION

Defects play an important role in high energy physics—see, e.g., Refs. [1–10] and references therein. In models described by real scalar fields, defect solutions are usually topological (kinklike) or nontopological (lumplike). In the present work we deal with models described by a single real scalar field, and our goal is to extend the deformation procedure introduced in Ref. [1] to new models, which support kinklike or lumplike solutions. To do this, in Sec. II we first consider the standard procedure. There we make the deformation prescription as general as possible, and we introduce new examples. Next, in Sec. III we implement two distinct extensions, one giving rise to a semivacuumless model and the corresponding domain wall, which serves as the seed for generation of dimension bubbles, as proposed in Refs. [2–4]. In the other extension we show how to implement deformations using nonbijective functions to deform models having lumplike solutions to generate new models that support kinklike solutions.

II. STANDARD PROCEDURE

We begin with a theory of a single real scalar field in (1,1) space-time dimensions. The Lagrangian density is as usual, and we use $V=V(\phi)$ to represent the potential that identifies the model. We also use the metric $(+,-)$, and we work with dimensionless fields and coordinates. The equation of motion for static fields is $d^2\phi/dx^2=V'(\phi)$, where the prime stands for the derivative with respect to the argument. We consider the broad class of potentials having at least one critical point $\bar{\phi}$ [that is, $V'(\bar{\phi})=0$], for which $V(\bar{\phi})=0$. In this case, solutions satisfying the conditions

$$\lim_{x\to-\infty}\phi(x)=\bar{\phi},\quad \lim_{x\to-\infty}\frac{d\phi}{dx}=0 \tag{1}$$

obey the first order equation (a first integral of the equation of motion) $(d\phi/dx)^2=2V[\phi(x)]$. For these solutions, the energy densities split into two equal parts of the gradient and potential energy densities.

Many important examples can be presented. The ϕ^4 model, with $V_4(\phi)=(1-\phi^2)^2/2$, is the prototype of theories having topological solitons (kinklike solutions) connecting two minima. In this case the solutions are $\phi(x)=\pm\tanh(x)$. A situation where nontopological (lumplike) solutions exist is the "inverted ϕ^4 model," with potential given by $V_{4i}(\phi)=\phi^2(1-\phi^2)/2$. In this case the lumplike defects are $\phi(x)=\pm\,\mathrm{sech}(x)$. One notes that the potential need not be nonnegative for all values of ϕ but the solution must be such that $V(\phi(x))\geqslant 0$ for the whole range $-\infty<x<+\infty$.

Both topological and nontopological solutions can be deformed, according to the prescription introduced in Ref. [1], to generate infinitely many new soluble problems. This method can be described in general form via the following statement. Let $f=f(\phi)$ be a bijective function having a continuous nonvanishing derivative. For each potential $V(\phi)$ bearing solutions satisfying conditions (1), the f-deformed model, defined by $\tilde{V}(\phi)=V[f(\phi)]/[f'(\phi)]^2$, possesses solutions given by $\tilde{\phi}(x)=f^{-1}(\phi(x))$, where $\phi(x)$ is a solution of the static equation of motion for the original potential $V(\phi)$.

We prove this assertion by noting that the static equation of motion of the new theory is written in terms of the old potential as

$$\frac{d^2\phi}{dx^2}=\frac{1}{f'(\phi)}V'[f(\phi)]-2V[f(\phi)]\frac{f''(\phi)}{[f'(\phi)]^3}. \tag{2}$$

On the other hand, taking the second derivative with respect to x of the deformed defect $\tilde{\phi}(x)$, one finds

$$\frac{d^2\tilde{\phi}}{dx^2}=\frac{1}{f'(\tilde{\phi})}\frac{d^2\phi}{dx^2}-\frac{f''(\tilde{\phi})}{[f'(\tilde{\phi})]^3}\left(\frac{d\phi}{dx}\right)^2. \tag{3}$$

It follows from the equation of motion and from $\tilde{\phi}(x)$ that $d^2\phi/dx^2=V'[f(\tilde{\phi})]$ and $(d\phi/dx)^2=2V[f(\tilde{\phi})]$ so that $\tilde{\phi}$ satisfies Eq. (2), as stated. The ratio between the energy density of the solution $\phi(x)$ of the undeformed model and the solution $\tilde{\phi}(x)$ of the f-deformed potential is $\varepsilon/\tilde{\varepsilon}=(df/d\phi)^2$.

Naturally, the deformation procedure heavily depends on the deformation function $f(\phi)$. Assume that f: $\mathbf{R}\to\mathbf{R}$ is bijective. In this case, the f deformation (and the deformation implemented by its inverse f^{-1}) can be applied successively and one can define equivalence classes of potentials related to each other by repeated applications of the f (or the f^{-1}) deformation. Each of such classes possesses an enumerable number of elements which correspond to smooth deformations of a representative one, all having the same topological characteristics. The generation sequence of new theories is depicted in the diagram below.

$$\begin{tikzcd}
\cdots & \hat{\hat{V}} \arrow[d] & \hat{V} \arrow[l, "f^{-1}"'] \arrow[d] & V \arrow[l, "f^{-1}"'] \arrow[r, "f"] \arrow[d] & \tilde{V} \arrow[r, "f"] \arrow[d] & \tilde{\tilde{V}} \arrow[d] & \cdots \\
\cdots & \hat{\hat{\phi}}_d & \hat{\phi}_d \arrow[l, "f"] & \phi_d \arrow[l, "f"] \arrow[r, "f^{-1}"'] & \tilde{\phi}_d \arrow[r, "f^{-1}"'] & \tilde{\tilde{\phi}}_d & \cdots
\end{tikzcd}$$

As an example not considered in Ref. [1], take the ϕ^6 model. This model, for which the potential $V_6(\phi)=\phi^2(1-\phi^2)^2/2$ has three degenerate minima at 0 and ± 1, is important since it allows the discussion of first-order transitions. It possesses kinklike solutions $\phi(x)=\pm\sqrt{[1\pm\tanh(x)]/2}$, connecting the central vacuum with the lateral ones. Take $f(\phi)=\sinh(\phi)$ as the deforming function. The sinh-deformed ϕ^6 potential is

$$\tilde{V}(\phi)=\frac{1}{2}\tanh^2(\phi)[1-\sinh^2(\phi)]^2 \tag{4}$$

and the sinh-deformed defects are

$$\tilde{\phi}(x)=\pm\,\mathrm{arcsinh}\sqrt{[1\pm\tanh(x)]/2}. \tag{5}$$

Notice that, since $f'(\phi)>1$ for the sinh deformation, the energy of the deformed solutions is diminished with respect to the undeformed kinks. The reverse situation emerges if one takes the inverse deformation implemented by $f^{-1}(\phi)=\mathrm{arcsinh}(\phi)$.

Interesting situations arise if one takes polynomial functions implementing the deformations. Consider $p_{2n+1}(\phi)=\Sigma_{j=0}^{n}c_j\phi^{2j+1}$, with $c_j>0$ for all $0\leqslant j\leqslant n$. These are bijective functions from **R** into **R** possessing positive derivatives. Fixing $n=0$ corresponds to a trivial rescaling of the field. For $n=1$, taking $c_0=c_1=1$, one has $f(\phi)=p_3(\phi)=\phi+\phi^3$ with the inverse given by $f^{-1}(\phi)=(2/\sqrt{3})\sinh[\mathrm{arcsinh}(3\sqrt{3}\phi/2)/3]$. Thus, the p_3-deformed ϕ^4 model, for which the potential has the form

$$\tilde{V}(\phi)=\frac{1}{2}\left(\frac{1-\phi^2-2\phi^4-\phi^6}{1+3\phi^2}\right)^2, \tag{6}$$

supports topological solitons given by

$$\tilde{\phi}_{\pm}(x)=\pm\frac{2}{\sqrt{3}}\sinh\left[\frac{1}{3}\mathrm{arcsinh}\left(\frac{3\sqrt{3}}{2}\tanh(x)\right)\right]. \tag{7}$$

Naturally, the inverse deformation can be implemented, leading to another new soluble problem. But if one takes $n\geqslant 2$, the inverse of p_{2n+1} cannot in general be expressed analytically in terms of known functions. This leads to circumstances where one knows analytically solutions of potentials which cannot be expressed in terms of known functions and, conversely, one has well-established potentials for which solitonic solutions exist but are not expressible in terms of known functions.

The procedure can also be applied to potentials presenting nontopological, lumplike, solutions which are of direct interest to tachyons [5]. Take, for example, the Lorentzian lump $\phi_l(x)=1/(x^2+1)$ which solves the equation of motion for the potential $V(\phi)=2(\phi^3-\phi^4)$ and satisfies conditions (1). Unlike the topological solitons, this kind of solution is not stable. In fact, the "secondary potential," which appears in the linearized Schrödinger-like equation satisfied by the small perturbations around $\phi_l(x)$ [6], is given by

$$U(x)=V''(\phi_l(x))=12\frac{x^2-1}{(x^2+1)^2}. \tag{8}$$

This potential is a symmetric volcanolike potential. It has zero mode given by $\eta_0(x)\sim\phi_l'(x)=-2x/(x^2+1)^2$, which does not correspond to the lowest energy state since it has a node. Deforming $V(\phi)=2(\phi^3-\phi^4)$ with $f(\phi)=\sinh(\phi)$ leads to the potential $\tilde{V}(\phi)=2\tanh^2(\phi)[\sinh(\phi)-\sinh^2(\phi)]$ which possesses the lumplike solution $\tilde{\phi}_l(x)=\mathrm{arcsinh}[1/(x^2+1)]$.

III. EXTENDED PROCEDURES

The deformation prescription is powerful. The conditions under which our procedure (see Ref. [1]) holds are maintained if we consider a function for which the contradomain is an interval of **R**, that is, if we take $f\colon \mathbf{R}\to I\subset\mathbf{R}$. In this case, however, the inverse transformation (engendered by $f^{-1}\colon I\to\mathbf{R}$) can be applied only for models where the values of ϕ are restricted to $I\subset\mathbf{R}$. We illustrate this possibility by asking for a deformation that leads to a model of the form needed in Ref. [3], described by a "semivacuumless" potential, in contrast with the vacuumless potential studied in Refs. [7,8]. Consider the new deformation function $f(\phi)=1-1/\sinh(e^{\phi})$, acting on the potential $V_4(\phi)=(1-\phi^2)^2/2$. The deformed potential is

$$\tilde{V}(\phi)=\frac{1}{2}e^{-2\phi}\,\mathrm{sech}^2(e^{\phi})[2\sinh(e^{\phi})-1]^2, \tag{9}$$

which is depicted in Fig. 1. The kinklike solution is

$$\tilde{\phi}(x)=\ln\left[\mathrm{arcsinh}\left(\frac{1}{1-\tanh(x)}\right)\right]. \tag{10}$$

The deformed potential (9) engenders the required profile: it has a minimum at $\bar{\phi}=\ln[\mathrm{arcsinh}(1/2)]$ and another one at $\phi\to\infty$. It is similar to the potential required in Ref. [3] for the existence of dimension bubbles. The bubble can be generated from the above (deformed) model, after removing the degeneracy between $\bar{\phi}$ and $\phi\to\infty$, in a way similar to the standard situation, which is usually implemented with the ϕ^4 potential, the undeformed potential that we used to generate Eq. (9). An issue here is that such a bubble is unstable against collapse, unless a mechanism is found to balance the

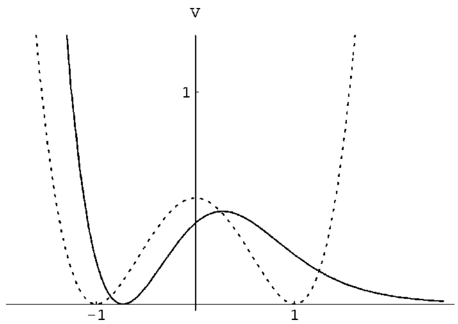

FIG. 1. The deformed potential $\widetilde{V}(\phi)$ of Eq. (9), plotted as a function of the scalar field ϕ; the dashed line shows the potential of the undeformed ϕ^4 model.

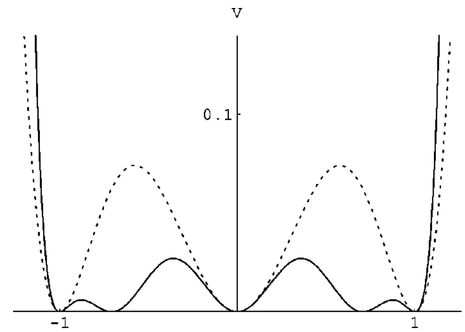

FIG. 2. The deformed potential $\widetilde{V}_6(\phi)$ and the undeformed ϕ^6 model (dashed line), plotted as functions of ϕ.

inward pressure due to the surface tension in the bubble. In Ref. [3], the mechanism used to stabilize the bubble requires another scalar field, in a way similar to the case of nontopological solitons previously proposed in Ref. [9]. This naturally leads to another scenario, which involves at least two real scalar fields.

The deformation procedure can be extended even further, by relaxing the requirement of f being a bijective function, under certain conditions. Suppose that f is not bijective but it is such that its inverse f^{-1} (which exists in the context of binary relations) is a multivalued function with all branches defined in the same interval $I \subset \mathbf{R}$. If the domain of definition of f^{-1} contains the interval where the values of the solutions $\phi(x)$ of the original potential vary, then $\widetilde{\phi}(x)=f^{-1}(\phi(x))$ are solutions of the new model obtained by implementing the deformation with f. However, one has to check whether the deformed potential $\widetilde{V}(\phi)=V[f(\phi)]/[f'(\phi)]^2$ is well defined on the critical points of f. In fact, this does not happen in general but occurs for some interesting cases.

Consider, for example, the function $f(\phi)=2\phi^2-1$; it is defined for all values of ϕ and its inverse is the double valued real function $f^{-1}(\phi)=\pm\sqrt{(1+\phi)/2}$, defined in the interval $[-1,\infty)$. If we deform the ϕ^4 model with this function we end up with the potential $\widetilde{V}(\phi)=\phi^2(1-\phi^2)^2/2$. The deformed kink solutions are given by $\widetilde{\phi}(x)=\pm\sqrt{[1+\phi(x)]/2}$ with $\phi(x)$ replaced by the solutions $[\pm\tanh(x)]$ of the ϕ^4 model, which reproduce the known solutions of the ϕ^6 theory. The important aspect, in the present case, is that the tanh kink corresponds to field values restricted to the interval $(-1,+1)$ which is contained within the domain of definition of the two branches of $f^{-1}(\phi)$. The fact that the ϕ^6 model can be obtained from the ϕ^4 potential in this way is interesting, since these models have distinct characteristics. Notice that the critical point of f at $\phi=0$ does not disturb the deformation in this case; this always occurs for potentials having a factor $(1-\phi^2)$, since the denominator of $\widetilde{V}(\phi)$ is canceled out. One can go on and apply this deformation to the ϕ^6 model; now, one finds the deformed potential $\widetilde{V}_6(\phi)=(1/2)\phi^2(1-\phi^2)^2(1-2\phi^2)^2$, with solutions given by

$$\widetilde{\phi}(x)=\pm\sqrt{1/2}\sqrt{1\pm\sqrt{[1\pm\tanh(x)]/2}}, \tag{11}$$

corresponding to kinks connecting neighboring minima (located at -1, $-1/\sqrt{2}$, 0, $1/\sqrt{2}$, and 1) of the potential, which is illustrated in Fig. 2. We can repeat the procedure for the potential $\widetilde{V}_6(\phi)$, to obtain a sequence of soluble polynomial potentials, all having exact kinklike solutions. This result should be contrasted with Ref. [10], which shows that it is in general hard to find solutions when the model includes higher-order powers in the scalar field.

The deformation implemented by the function $f(\phi)=2\phi^2-1$ can also be applied to a potential possessing lumplike solutions. Consider the inverted ϕ^4 potential $V_{4i}(\phi)=\phi^2(1-\phi^2)/2$, which has the lump solutions $\phi(x)=\pm\,\mathrm{sech}(x)$. The deformed potential, in this case, is given by $\widetilde{V}_{4i}(\phi)=(1/2)(1-\phi^2)(\phi^2-1/2)^2$. This potential, which is also unbounded from below, vanishes for $\phi=\pm1/\sqrt{2}$, ±1, has an absolute maximum at $\phi=0$, and local minima and maxima for $\pm1/\sqrt{2}$ and $\pm\sqrt{5/6}$, respectively. Figure 3 shows a plot of this potential. Again, the number of solutions duplicates using such a deformation: there are two solutions, $\widetilde{\phi}(x)_l^{(\pm)}=\pm\sqrt{[1+\mathrm{sech}(x)]/2}$, which correspond to lumps running between the local minima and the lateral zeros of the potential, and also

$$\widetilde{\phi}(x)_k^{(\pm)}=\begin{cases}\mp\sqrt{[1-\mathrm{sech}(x)]/2}, & x\leqslant 0,\\ \pm\sqrt{[1-\mathrm{sech}(x)]/2}, & x\geqslant 0,\end{cases} \tag{12}$$

which correspond to kinklike solutions connecting the minima $\pm1/\sqrt{2}$. This is a very unusual example where nontopological or lumplike solutions are deformed into topological or kinklike solutions.

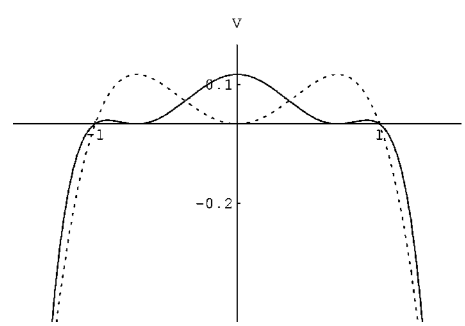

FIG. 3. The deformed potential $\widetilde{V}_{i4}(\phi)$ and the undeformed inverted ϕ^4 model (dashed line), plotted as functions of ϕ.

Potentials that have a factor $(1-\phi^2)$ can also be deformed using the function $f(\phi)=\sin(\phi)$, producing many interesting situations. In fact, suppose the potential can be written in the form $V(\phi)=(1-\phi^2)U(\phi)$. This is always possible for all well-behaved potentials that vanish at both values $\phi=\pm 1$, as shown by Taylor expansion. Then, the sine deformation leads to the potential $\tilde{V}(\phi)=U[\sin(\phi)]$, which is a periodic potential, the critical points of $\sin(\phi)$ not causing any problem to the deformation process. The inverse of the sine function is the infinitely valued function $f^{-1}(\phi)=(-1)^k\arcsin(\phi)+k\pi$, with $k\in\mathbf{Z}$ and $\arcsin(\phi)$ being the first determination of $\arcsin(\phi)$ (which varies from $-\pi/2$, for $\phi=-1$, to $+\pi/2$, when $\phi=+1$), defined in the interval $(-1,+1)$. So for each solution of the original potential, whose field values range in the interval $(-1,+1)$, one finds infinitely many solutions of the deformed, periodic, potential.

Consider first the ϕ^4 model. Applying the sine deformation to it, one gets $\tilde{V}(\phi)=\cos^2(\phi)/2$, which is one of the forms of the sine-Gordon potential. The deformed solution thus obtained is given by $\tilde{\phi}(x)=(-1)^k\arcsin[\pm\tanh(x)]+k\pi$, which correspond to all the kink solutions (connecting neighboring minima) of this sine-Gordon model. For example, the kink solutions $\pm\tanh x$, which connect the minima $\phi=\pm 1$ of the ϕ^4 model in both directions, are deformed into the kinks $\pm\arcsin[\tanh(x)]=2\arctan(e^{\pm x})-\pi/2$ (which run between $-\pi/2$ and $\pi/2$) if one takes $k=0$ while, for $k=1$, the resulting solutions connect the minima $\pi/2$ and $3\pi/2$ of the deformed potential.

This example can be readily extended to other polynomial potentials, leading to a large class of sine-Gordon type of potentials. For instance, the ϕ^6 model, $V(\phi)=\phi^2(1-\phi^2)^2/2$, deformed by the sine function, becomes the potential $\tilde{V}(\phi)=(1/2)\cos^2(\phi)[1-\cos^2(\phi)]$, which has kinklike solutions given by

$$\tilde{\phi}(x)=\pm(-1)^k\arcsin\sqrt{[1\pm\tanh(x)]/2}+k\pi. \tag{13}$$

On the other hand, if one considers $V(\phi)=(1-\phi^2)^3/2$, which is unbounded below and supports kinklike solutions connecting the two inflection points at ± 1, one gets the potential $\tilde{V}(\phi)=(1/2)\cos^4(\phi)$, which is solved by $\tilde{\phi}(x)=\pm(-1)^k\arcsin(x/\sqrt{1+x^2})+k\pi$.

Another particularly interesting situation where nontopological solutions are deformed into topological solutions appears if one considers the inverted ϕ^4 model, which presents lumplike solutions. The sine deformation of the potential $V(\phi)=\phi^2(1-\phi^2)/2$ leads to the potential $\tilde{V}(\phi)=\sin^2(\phi)/2$. In this case, the lump solutions of $V(\phi)$, namely, $\phi(x)=\pm\mathrm{sech}(x)$, are deformed into $\tilde{\phi}(x)=\pm(-1)^k\arcsin[\mathrm{sech}(x)]+k\pi$. Consider the $(+)$ solution and take initially $k=0$. As x varies from $-\infty$ to 0, $\mathrm{sech}(x)$ goes from 0 to 1, and $\arcsin[\mathrm{sech}(x)]=2\arctan(e^x)$ changes from 0 to $\pi/2$. If one continuously makes x go from 0 to $+\infty$, then the deformed solution passes to the $k=1$ branch of $\arcsin(\phi)$, $-\arcsin[\mathrm{sech}(x)]+\pi$ $[=2\arctan(e^x)$ for $0\leqslant x<+\infty]$, which varies from $\pi/2$ to π as x goes from 0 to $+\infty$. Thus, in this case, the lump solution $+\mathrm{sech}(x)$ of the inverted ϕ^4 model is deformed into the kink of the sine-Gordon model connecting the minima $\phi=0$ and $\phi=\pi$. Under reversed conditions (taking the $k=1$ branch before the $k=0$ one), the lump solution $-\mathrm{sech}(x)$ leads to the antikink solution of the sine-Gordon model running from the minimum $\phi=\pi$ to 0. The other topological solutions of the sine-Gordon model are obtained by considering the other adjacent branches of $\arcsin(\phi)$.

IV. COMMENTS AND CONCLUSIONS

In previous work on deformed defects [1], we stressed that the deformation procedure strongly depends on a function $f=f(\phi)$, the deformation function, and there we considered only bijective functions that obey f: $\mathbf{R}\to\mathbf{R}$. In the present work, we have extended the deformation procedure with the inclusion of two new possibilities. First, we have considered deformation functions such that f: $\mathbf{R}\to I$ with $I\subset\mathbf{R}$, which gives rise to new models such as the one recently considered in Ref. [3], requiring a semivacuumless potential. Furthermore, we have shown how to deal with nonbijective functions to build new models. This last case leads to very interesting possibilities for deforming models that support nontopological defects, to give rise to models that support topological defects.

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