Logica universalis 1 (2007), 243–276
1661-8297/020243-34, DOI 10.1007/s11787-007-0013-z

Logica Universalis

Abstract Logics, Logic Maps, and Logic Homomorphisms

Steffen Lewitzka

Abstract. What is a logic? Which properties are preserved by maps between logics? What is the right notion for equivalence of logics? In order to give satisfactory answers we generalize and further develop the topological approach of [4] and present the foundations of a general theory of abstract logics which is based on the abstract concept of a theory. Each abstract logic determines a topology on the set of theories. We develop a theory of logic maps and show in what way they induce (continuous, open) functions on the corresponding topological spaces. We also establish connections to well-known notions such as translations of logics and the satisfaction axiom of institutions [5]. Logic homomorphisms are maps that behave in some sense like continuous functions and preserve more topological structure than logic maps in general. We introduce the notion of a logic isomorphism as a (not necessarily bijective) function on the sets of formulas that induces a homeomorphism between the respective topological spaces and gives rise to an equivalence relation on abstract logics. Therefore, we propose logic isomorphisms as an adequate and precise notion for equivalence of logics. Finally, we compare this concept with another recent proposal presented in [2].

Mathematics Subject Classification (2000). Primary 03B22; Secondary 54A99, 03G30.

Keywords. Abstract logics, logic maps, logic translations, logic homomorphism, universal logic, topology, institutions.

1. Introduction

Logics that abstract from model theory are often given as a deduction system $\mathcal{L} = (L, \Vdash)$, where L is some set of formulas and $\Vdash$ is a deduction (or consequence) relation satisfying usually the three Tarski axioms of extensiveness, monotonicity and idempotence. These axioms are equivalent to the following:

This research was supported by the grant CNPq/FAPESB 350092/2006-0.

(i) If $a \in A$, then $A \Vdash a$.
(ii) If $A \Vdash b$ for all $b \in B$, and if $B \Vdash c$, then $A \Vdash c$.

These rather intuitive axioms allow to capture a huge class of concrete logics (given as a deduction system). (One exception, however, is given by non-monotonic logics, which play an important role in artificial intelligence.) It is certainly not wrong to claim that logic as a traditional area of research can be described as the science of sound reasoning. In this sense, such a kind of deduction system then may serve as a rather general concept of logic. However, an essential aspect is neglected in this approach, namely the interplay between syntax and semantics, that is, model theory. In fact, one problem of deduction systems is the absence of an adequate notion of consistency. Let us say that a notion of consistency is adequate if it coincides with model-theoretic satisfaction. In order to give a generic concept of consistency in a deduction system one usually defines:

A set A of formulas is consistent, if there is some formula a such that $A \nVdash a$. Otherwise, A is inconsistent.

In particular, the whole set L of formulas is, by this definition, always inconsistent. But there are logics where L is satisfiable by some model. One example is the 3-valued paraconsistent logic LP, as discussed in [7]. We quote another example, which was already sketched in [8]: assume first order equational logic over some given signature Σ (the set of formulas consists of all equations over Σ) and consider a Σ-structure $\mathcal{A}$ with exactly one element. Then $\mathcal{A}$ is a model of the set of all formulas L, since it satisfies all equations of L. We conclude that not for all logics, given as a deduction system, the usual notion of consistency is adequate.

One may find further examples that illustrate the relevance of the semantical component. For instance, it might be possible that two first order logics have the same consequence relation (thus, they are equal as deduction systems) but in one logic we admit models of arbitrary cardinality and in the other one we only admit models of a bounded infinite cardinality. Then these logics may have different behaviors with respect to some classical model-theoretic meta-theorems. The general problem here is that not all semantical properties can be expressed syntactically (recall, for example, that two models of different infinite cardinality can be elementary equivalent in first order logic, however, they are not isomorphic).

For these reasons one might prefer a concept of logic that includes the model-theoretic component (see [4, 5]). However, for many purposes it is not necessary to consider all specific semantical properties, such as cardinality questions of (infinite) models. In these cases a deduction system with an adequate notion of consistency is sufficient in order to express many interesting properties of a logic. Since we have seen that a pure deduction system may lead to a counterintuitive (inadequate) notion of consistency, we adapt some ideas of [8] to our approach and choose the abstract notion of a theory as the basic component of a logic. A theory is a priori a consistent set of formulas. It will follow from our definition of abstract logic that theories are exactly the consistent, deductively closed sets. There are two important advantages of this approach to abstract logics: it captures in an abstract way model-theoretic logics as well as logics based on any proof-theoretic system (of course, both types of logics generate theories, i.e., consistent and deductively closed sets), and it yields an adequate notion of consistency.

Abstraction in our approach refers also to another thing: we do not require any syntactical assumptions, the expressions (formulas) of an abstract logic are in principle pure abstract objects without any inner structure.

An abstract logic $\mathcal{L} = (Expr_{\mathcal{L}}, Th_{\mathcal{L}})$ is given by a set of expressions (or formulas) and a subset $Th_{\mathcal{L}}$ of the powerset of $Expr_{\mathcal{L}}$. $Th_{\mathcal{L}}$ is the set of theories. Now we may define consistency, validity and similar notions by means of the set of theories (e.g., a set of formulas is consistent, if it is contained in some theory). In particular, the consequence relation is determined in the usual way. One can see that the resulting consequence relation satisfies the Tarski axioms, i.e., the axioms (i) and (ii) above. Moreover, we require two intersection conditions as axioms in our definition of abstract logics:

- Any intersection of a nonempty set of theories is again a theory;
- If the empty set $\varnothing$ is a theory, then $\varnothing$ is the intersection of a set of non-empty theories.

The first condition will guarantee that the set of theories is exactly the set of all consistent and deductively closed sets of formulas. The second condition will be technically useful in order to deal with our logic maps. It also expresses the following intuition: if there is some formula contained in all (non-empty) theories, then this formula is valid. In this case, the empty set should not be a theory.

The above mentioned axioms are not considered in [8]. This may lead to counterintuitive consequences: two logics may have the same consequence relation although there is no bijection between the respective sets of theories. Thus, the logics can not be considered as equivalent or equal in the topological sense (that is, there is no homeomorphism between the respective topological spaces). In fact, the topological potential, as explored in the present paper (or in [4]), remains unnoticed in [8]. In [8] there are also defined mappings between logics and it is shown that they preserve some properties such as consequence and consistency. Our concept of a logic map is based on that notion given in [8] and is adapted to our approach. However, in [8] these mappings are not studied in more detail. We aim in this paper to develop a theory of logic maps in order to find conditions under which they induce (continuous, open) functions between the respective topological spaces.

The starting point of the present research is the observation that each abstract logic induces a topology on the set of theories. For example, if the abstract logic is finitary and has classical negation and conjunction, then the set of maximal theories forms a boolean space (i.e., a topological space which is Hausdorff, zero-dimensional and compact). Such a characterization provides a topological invariant for this specific class of abstract logics. (One may expect that many other classes of abstract logics also give rise to such topological invariants.) Then we define logic maps as functions (on the set of expressions) that preserve in some sense the structure of the theories. We study extensively logic maps and prove that they are in general stronger than translations. However, both concepts coincide in many cases (Proposition 3.3). One of the most interesting properties of logic maps is that under the normality condition (Definition 3.4) they give rise to functions, called complements, on the theory spaces of the respective logics. After studying properties of logic maps and their complements we show in Corollary 3.15 some conditions under which complements of logic maps are continuous or open functions. Thus, we reveal a connection between logic maps (translations) and continuous and open functions between the respective topological spaces. Furthermore, we introduce a function which is given in a unique way by a logic map h. This function, called the inverse complement of h, is always a continuous map from the target to the source space (Proposition 3.21). If h has exactly one complement, then the inverse complement is in fact the inverse function of the complement. What is interesting here is that the inverse complement together with its logic map satisfies an equation which corresponds to the satisfaction axiom of institutions (see [5]). This is expressed by Corollary 3.23. However, the connections between abstract logics and institutions remain to be further investigated.

In Section 4 we introduce logic homomorphisms of abstract logics. Logic homomorphisms are logic maps that behave in some sense as continuous functions: they send basic open sets of the target space to open sets of the source space. In fact, in Proposition 4.6 we show that in many cases logic homomorphisms lead directly to open and continuous maps between the respective spaces. Logic homomorphisms are strictly stronger than logic maps (and translations). In general, they preserve more topological structure and expressive power of a logic than (normal) logic maps. This is proved by the Counter Example 8: we are able to present two logics so that there is a normal logic map but no logic homomorphism between them.

Furthermore, we study strong logic homomorphisms. Strong homomorphisms are interesting for the following reason: the existence of a strong logic homomorphism, which is not necessarily a logic isomorphism, implies an homeomorphism between the respective topological spaces (Corollary 4.10). In Example 9 we present a strong logic homomorphism which is not a logic isomorphism.

Logic isomorphisms are L-surjective normal logic homomorphisms. In contrast to an earlier version given in [4] we no longer require a logic isomorphism to be a bijective function on formulas. It is sufficient to require that it is bijective on the respective equivalence classes of formulas modulo logical equivalence. We agree here with arguments given in [2] defending that bijections on formulas are too strong for a suitable concept of equivalence of logics. On the other hand, our notion of logic isomorphism is strong enough to induce a homeomorphism between the respective spaces. Furthermore, we show that this notion behaves well in the sense that the relation "isomorphic" between logics is an equivalence relation (Theorems 4.15 and 4.16). Therefore we propose the notion of logic isomorphism as an adequate concept of equivalence of abstract logics. Finally, we show that this concept coincides with the notion of "equipollence of logical systems" introduced in [2], if we assume the following restrictions:

An abstract logic must satisfy the syntactical rules given in [2], our logic maps are uniform translations (or logical system morphisms in the sense of [2]) and the abstract logics in consideration are either all regular or are all singular.

Perhaps the second restriction is the strongest one, since many translations relevant in practice are not uniform.

As far as we know, the topological approach to logic maps developed here (and in the previous paper [4]) is new. However, in the literature one can find works on logic translations borrowing notions from topology (such as "continuous", "homeomorphic" or "topological") in order to describe properties of translations, see, for instance, [6] or [3]. One should be aware that in these cases there is usually no underlying topological space defined. Of course, these notions are used in order to establish a certain analogy to some concepts and situations from general topology. Instead of a true topological space one works with a closure space given by the consequence relation. Since this closure operator does not satisfy all the Kuratowski closure axioms characteristic of a topological space, the resulting "continuous mappings" or "topological" properties are not true continuous functions or topological properties in the sense of general topology (this was also pointed out in [6]). In fact, our Example 8 illustrates that there may exist a "continuous mapping" (i.e., a logic translation in the sense of [3], Definition 2.3), which does not induce a (true) continuous map between the respective underlying topological spaces.

Some of the concepts and results elaborated in this paper are based on the work [4], which can be seen as an early stage of the current research. In [4] model-theoretical abstract logics were considered, which now can be treated inside the more general setting of abstract logics. We were able to generalize (and to simplify) in a considerable way many notions and results of [4] in this broader context. In particular, relationships between logics (logic maps and logic homomorphisms) can be defined in a more flexible and general way than in the restricted context of [4].

2. Abstract logics and topology

Definition 2.1. An abstract logic $\mathcal{L}$ is a pair $\mathcal{L} = (Expr_{\mathcal{L}}, Th_{\mathcal{L}})$, where $Expr_{\mathcal{L}}$ is a set of expressions (or formulas) and $Th_{\mathcal{L}}$ is a subset of the power set of $Expr_{\mathcal{L}}$, called the set of theories, such that the following axioms are satisfied:

(i) If $\mathcal{T} \subseteq Th_{\mathcal{L}}$ and $\mathcal{T} \neq \varnothing$, then $\cap\mathcal{T} \in Th_{\mathcal{L}}$.
(ii) If $\varnothing \in Th_{\mathcal{L}}$, then there is a set of theories $\mathcal{T} \subseteq Th_{\mathcal{L}}$ such that $\varnothing \notin \mathcal{T}$ and $\varnothing = \cap\mathcal{T}$.

The abstract logic $\mathcal{L}' = (Expr_{\mathcal{L}'}, Th_{\mathcal{L}'})$ is a sublogic of the abstract logic $\mathcal{L} = (Expr_{\mathcal{L}}, Th_{\mathcal{L}})$, written $\mathcal{L}' \subseteq \mathcal{L}$, if $Expr_{\mathcal{L}'} = Expr_{\mathcal{L}}$ and $Th_{\mathcal{L}'} \subseteq Th_{\mathcal{L}}$.

We say that the logic $\mathcal{L}$ is trivial, if $Th_{\mathcal{L}} = \varnothing$. $\mathcal{L}$ is regular, if $Expr_{\mathcal{L}} \notin Th_{\mathcal{L}}$. If $\mathcal{L}$ is not regular, then we say that $\mathcal{L}$ is singular. The empty logic is the abstract logic with the empty set of expressions.

Axiom (i) is rather natural. It will guarantee that the set of theories is exactly the set of all consistent and deductively closed sets. Axiom (ii) says that the empty set can be a theory only if it is the intersection of nonempty theories. In other words, the empty set is a theory if and only if it is generated (by means of intersection) by other theories. This leads us to the notion of generators:

Definition 2.2. (i) We say that a subset $\mathcal{G} \subseteq Th_{\mathcal{L}}$ is a generator set, if for each theory $T \in Th_{\mathcal{L}}$, $T = \cap\mathcal{T}$, for some $\mathcal{T} \subseteq \mathcal{G}$. A generator set is minimal, if any proper subset is not a generator set.

(ii) A theory $T \in Th_{\mathcal{L}}$ is called prime, if T is not the intersection of other theories. That is, the theory T is prime if whenever $T = \cap\mathcal{T}$, then $T \in \mathcal{T}$ or $\mathcal{T} = \varnothing$.[1]

The set of all prime theories is denoted by $PTh_{\mathcal{L}}$.

Axiom (ii) of the definition of abstract logics says that the empty set - if it is a theory - can not be prime. Furthermore, no minimal generator set countains the empty theory as an element. In some sense one could say that prime theories are those theories which are not generated (by means of intersections) - they are given a priori. Thus, a prime theory must be contained in any generator set.

Logics with model-theoretic semantics give natural examples of abstract logics:

Suppose that L is a set of well-formed formulas, C is a class of models or interpretations and $\vDash$ is a satisfaction relation between models of C and L-expressions. For $M \in C$ define $Th(M) := \{a \in L \mid M \vDash a\}$, the theory of M. Then L together with the set of all intersections of nonempty subsets of $\{Th(M) \mid M \in C\}$ forms an abstract logic. Clearly, $\{Th(M) \mid M \in C\}$ is a generator set of this abstract logic. We say that the logic is generated by the class of models C, or that C generates this abstract logic.

We sketch out three concrete examples:

Example 1. Classical first order logic over a given signature Σ. The expressions are first order formulas over Σ. The satisfaction relation is defined in the usual way. Then the logic $\mathcal{L}$ generated by the class of all Σ-structures can be considered as a (boolean) abstract logic. Every subclass of Σ-structures generates a sublogic (which is still classical but, in general, no longer boolean, since compactness may fail).[2]

Example 2. Intuitionistic propositional logic (over an infinite set of propositions P). The expressions are inductively defined in the usual way over an infinite set of propositional variables P. We assume that there are the usual connectivities $\rightarrow$, $\vee$, $\wedge$ and $\sim$. We consider Kripke semantics (possible world semantics): An

[1] The latter case, $\mathcal{T} = \varnothing$, means that $T = Expr_{\mathcal{L}} = \cap\varnothing$.

[2] For the anticipated concepts of "boolean" and "classical abstract logic" see Definition 2.5 below.

intuitionistic frame is a pair $F = (W, \leq)$ with a nonempty set W and a partial ordering $\leq$ on W. The elements of W are called possible worlds. An intuitionistic variable assignment is a map $v : P \to Pow(W)$ such that $w \leq w'$ and $w \in v(p)$ implies $w' \in v(p)$.[3] If F is an intuitionistic frame, w a possible world and v an intuitionistic variable assignment, then we call the triple $I = (F, v, w)$ an intuitionistic interpretation. The satisfaction relation $\vDash$ is inductively defined in the following way:

$$I \vDash p :\Longleftrightarrow w \in v(p)$$
$$I \vDash a \vee b :\Longleftrightarrow I \vDash a \text{ or } I \vDash b$$
$$I \vDash a \wedge b :\Longleftrightarrow I \vDash a \text{ and } I \vDash b$$
$$I \vDash \sim a :\Longleftrightarrow (F, v, w') \nvDash a, \text{ for all } w' \in W \text{ with } w \leq w'$$
$$I \vDash a \to b :\Longleftrightarrow (F, v, w') \nvDash a \text{ or } (F, v, w') \vDash b, \text{ for all } w' \in W \text{ with } w \leq w' .$$

Now, the class of all intuitionistic interpretations $I = (F, v, w)$, considered as models, generates an abstract logic which we call intuitionistic propositional logic over the set of propositions P.

Example 3. Classical propositional logic (over an infinite set of propositions P). We consider the same set of formulas and Kripke semantics as for intuitionistic propositional logic. If the set W of an intuitionistic frame $F = (W, \leq)$ has exactly one element, then the intuitionistic variable assignment v can be seen as a function $v : P \to \{0, 1\}$. In this case the frame $F = (W, \leq)$ and the intuitionistic interpretation $I = (F, v, w)$ are in some sense trivial and depend only on the variable assignment v. Therefore, we may look at this variable assignment as an interpretation, i.e., a model. Now we consider the abstract sublogic of intuitionistic propositional logic which is generated by all such "trivial" intuitionistic interpretations $I = (F, v, w)$, that is, by all variale assignments $v \in 2^P$, viewed as models. We call this abstract logic classical propositional logic over the set of propositions P.

Remark 2.3. Sometimes we look at theories as abstract models in the following sense: Let $\mathcal{L}$ be an abstract logic, $T \in Th_{\mathcal{L}}$ a theory and $a \in Expr_{\mathcal{L}}$ an expression. Then we call T an abstract model of a and write $T \vDash_{\mathcal{L}} a$, if $a \in T$. For a set A of $\mathcal{L}$-expressions we say that T is an abstract model of A and write $T \vDash_{\mathcal{L}} A$, if $T \vDash_{\mathcal{L}} a$ for all $a \in A$ (i.e., if $A \subseteq T$).

Most of the relevant concepts that we will need are derivable from the fundamental concept of a theory:

Definition 2.4. Every abstract logic $\mathcal{L}$ gives rise to a consequence relation $\Vdash_{\mathcal{L}} \subseteq Pow(Expr_{\mathcal{L}}) \times Pow(Expr_{\mathcal{L}})$ defined by

$$A \Vdash_{\mathcal{L}} B :\Longleftrightarrow B \subseteq \cap\{T \in Th_{\mathcal{L}} \mid A \subseteq T\} .$$

[3] $Pow(A)$ denotes the power set of A, i.e., the set of all subsets of A.

If $B = \{b\}$, then we write $A \Vdash_{\mathcal{L}} b$ instead of $A \Vdash_{\mathcal{L}} B$.[4]

A set $A \subseteq Expr_{\mathcal{L}}$ is deductively closed, if $A \Vdash_{\mathcal{L}} a$ implies $a \in A$. The deductive closure of A is defined by $A^{\Vdash_{\mathcal{L}}} := \{a \in Expr_{\mathcal{L}} \mid A \Vdash_{\mathcal{L}} a\}$.

Two sets of expressions A and B are $\mathcal{L}$-equivalent, if $A \Vdash_{\mathcal{L}} B$ and $B \Vdash_{\mathcal{L}} A$. If A, B are $\mathcal{L}$-equivalent, then we write $A =_{\mathcal{L}} B$. If $A = \{a\}$ and $A =_{\mathcal{L}} B$, then we write $a =_{\mathcal{L}} B$ (and analogously for the case $B = \{b\}$).

Note that $A \Vdash_{\mathcal{L}} a$ holds if and only if $T \vDash_{\mathcal{L}} a$ whenever $T \vDash_{\mathcal{L}} A$, for every theory $T \in Th_{\mathcal{L}}$ (where we interpret theories as abstract models in the sense explained above.) It is easy to see that $\Vdash_{\mathcal{L}}$ is a closure operator, i.e., it is mononotic, idempotent and reflexive (extensive). That is, $\Vdash_{\mathcal{L}}$ satisfies exactly the above discussed Tarski axioms of a consequence relation.

Notice also that $\Vdash_{\mathcal{L}'} \supseteq \Vdash_{\mathcal{L}}$, if $\mathcal{L}' \subseteq \mathcal{L}$.

Definition 2.5. Let $\mathcal{L}$ be an abstract logic.

(i) A set $A \subseteq Expr_{\mathcal{L}}$ of expressions is consistent, if $A \subseteq T$ for some $T \in Th_{\mathcal{L}}$. Otherwise, A is inconsistent.[5]

(ii) A set $A \subseteq Expr_{\mathcal{L}}$ is maximally consistent, if it is consistent and for any $a \in Expr_{\mathcal{L}} \smallsetminus A$ the set $A \cup \{a\}$ is inconsistent.[6] We denote the set of maximal theories (=the set of maximally consistent sets) by $MTh_{\mathcal{L}}$.

(iii) $\mathcal{L}$ is called finitary, if for all $A \subseteq Expr_{\mathcal{L}}$ the following holds: A is consistent if and only if every finite subset of A is consistent.

(iv) The consequence relation $\Vdash_{\mathcal{L}}$ is finitary, if for all sets $A \subseteq Expr_{\mathcal{L}}$ and all expressions $a \in Expr_{\mathcal{L}}$ the following holds: If $A \Vdash_{\mathcal{L}} a$, then there is some finite subset $A_f \subseteq A$ such that $A_f \Vdash_{\mathcal{L}} a$.

(v) $\mathcal{L}$ has classical negation, if $MTh_{\mathcal{L}}$ is a generator set and for each expression b there is some expression c such that for all maximal theories T the equivalence $b \in T \Longleftrightarrow c \notin T$ holds. Such an expression c is called a classical negation of b. If there is some operator $\sim$ that assigns to each expression $b \in Expr_{\mathcal{L}}$ a classical negation c of b, then we denote c by $\sim b$ and say that $\sim$ is a classical negation of logic $\mathcal{L}$.

(vi) $\mathcal{L}$ has (finite) conjunction, if for every pair of expressions b, c there is some expression d such that for all $T \in Th_{\mathcal{L}}$ the equivalence $\{b, c\} \subseteq T \Longleftrightarrow d \in T$ holds. d is called a conjunction of the expressions b and c. If there is a operator $\wedge$ which assigns to each pair of expressions b, c a conjunction d, then we write $b \wedge c$ for d and say that $\wedge$ is a conjunction of logic $\mathcal{L}$.

[4] Note that $A \Vdash_{\mathcal{L}} B \Longleftrightarrow A \Vdash_{\mathcal{L}} b$, for all $b \in B$.

[5] Notice the following: If A is inconsistent, then $A \Vdash_{\mathcal{L}} a$ for all $a \in Expr_{\mathcal{L}}$, since by a set-theoretic convention $\cap\{T \in Th_{\mathcal{L}} \mid A \subseteq T\} = \cap\varnothing = Expr_{\mathcal{L}}$. A trivial logic has no consistent set. A regular logic has some inconsistent set. In a singular logic every set of expressions is consistent, that is, there is no inconsistent set. If $\mathcal{L}$ is regular and A is consistent, then there is some $a \in Expr_{\mathcal{L}}$ such that $A \nVdash_{\mathcal{L}} a$.

[6] Clearly, a maximally consistent set is a maximal theory, i.e., a theory that is maximal with respect to set-theoretic inclusion.

(vii) An abstract logic with finite conjunction and classical negation is called a classical (abstract) logic. A boolean logic is a finitary classical abstract logic.

Remark 2.6. Let $\mathcal{L}$ be an abstract logic. Every maximal theory is a prime theory. A prime theory is contained in every generator set. Thus, if $G_{\mathcal{L}}$ is any generator set, then $MTh_{\mathcal{L}} \subseteq PTh_{\mathcal{L}} \subseteq G_{\mathcal{L}}$. In particular, if $MTh_{\mathcal{L}}$ ($PTh_{\mathcal{L}}$) is a generator set, then $MTh_{\mathcal{L}}$ ($PTh_{\mathcal{L}}$) is also the minimal generator set, respectively. Moreover, if $MTh_{\mathcal{L}}$ is a generator set, then $MTh_{\mathcal{L}} = PTh_{\mathcal{L}}$.

Proof. Let $T \in MTh_{\mathcal{L}}$. Since T is a maximal theory, it can not be the intersection of a set of theories, all distinct from T. Thus, T must be prime. Clearly, a prime theory must be contained in any generator set. Now the remaining assertions follow immediately. □

Lemma 2.7. *Let $\mathcal{L}$ be an abstract logic. A set $T \subseteq Expr_{\mathcal{L}}$ is a theory if and only if T is consistent and deductively closed.*

Proof. If T is a theory, then T is consistent by definition. From the definition of the consequence relation it follows easily that T is deductively closed. Now suppose that T is consistent and deductively closed. It follows that T is the intersection of all theories containing T. The first axiom of the definition of abstract logics says that T is a theory. □

Definition 2.8. Let $\mathcal{L} = (Expr_{\mathcal{L}}, Th_{\mathcal{L}})$ be an abstract logic and let $\mathcal{T} \subseteq Th_{\mathcal{L}}$ be any set of theories. For $a \in Expr_{\mathcal{L}}$ we define

$$a^{\mathcal{T}} := \{T \in \mathcal{T} \mid a \in T\}\,.$$

Analogously, for $A \subseteq Expr_{\mathcal{L}}$ we put

$$A^{\mathcal{T}} := \{T \in \mathcal{T} \mid A \subseteq T\}\,.$$

Furthermore, we define

$$\mathcal{S}(\mathcal{T}) := \begin{cases} \{a^{\mathcal{T}} \mid a \in Expr_{\mathcal{L}}\} \cup \{\{\varnothing\}\}\,, & \text{if } \varnothing \in \mathcal{T} \\ \{a^{\mathcal{T}} \mid a \in Expr_{\mathcal{L}}\}\,, & \text{else}\ . \end{cases}$$

and

$$\mathcal{B}(\mathcal{T}) := \big\{\sigma \mid \sigma \text{ is a finite non-empty intersection of elements of } \mathcal{S}(\mathcal{T})\big\}\,,$$

that is, $\sigma \in \mathcal{B}(\mathcal{T}) \Longleftrightarrow \sigma = \rho_1 \cap \cdots \cap \rho_n$, for some $n \geq 1$ and some $\rho_1, \ldots, \rho_n \in \mathcal{S}(\mathcal{T})$.

We call the set $\mathcal{S}(\mathcal{T})$ ($\mathcal{B}(\mathcal{T})$) the subbasis (basis) of the space $\mathcal{T}$, respectively. If $\mathcal{T} = Th_{\mathcal{L}}$, then we say that $\mathcal{B}_{\mathcal{L}} := \mathcal{B}(\mathcal{T}) = \mathcal{B}(Th_{\mathcal{L}})$ is the basis of the logic $\mathcal{L}$ and $\mathcal{S}_{\mathcal{L}} := \mathcal{S}(Th_{\mathcal{L}})$ is the subbasis of $\mathcal{L}$. In this case we write $a^{*_{\mathcal{L}}}$ ($A^{*_{\mathcal{L}}}$) instead of $a^{Th_{\mathcal{L}}}$ ($A^{Th_{\mathcal{L}}}$), respectively.

Note that $A^{*_{\mathcal{L}}} = \cap\{a^{*_{\mathcal{L}}} \mid a \in A\}$.

There is a justification for the above defined terminology:

Proposition 2.9. *Let $\mathcal{L}$ be an abstract logic.*

(i) *The set* $\mathcal{S}_\mathcal{L} = \mathcal{S}(Th_\mathcal{L})$ *is a subbasis of a topology* $\sigma_\mathcal{L}$ *on* $Th_\mathcal{L}$. *The basis of the logic* $\mathcal{L}$, $\mathcal{B}_\mathcal{L} = \mathcal{B}(Th_\mathcal{L})$, *is a basis of the topology* $\sigma_\mathcal{L}$. *In other words,* $(Th_\mathcal{L}, \sigma_\mathcal{L})$ *forms a topological space where the elements of* $\mathcal{B}_\mathcal{L}$ *are the basic open sets.*

(ii) *Now let us assume the following: if* $\mathcal{L}$ *has no valid formula (that is,* $\varnothing \in Th_\mathcal{L}$ *), then* $\mathcal{L}$ *has a inconsistent formula* $\bot$ *(that is,* $\bot^{*_\mathcal{L}} = \varnothing$ *). Let* $G \subseteq Th_\mathcal{L}$ *be a set of generators and let* $\mathcal{S}(G)$ *be the subbasis of the subspace* G. *Then the following conditions are equivalent:*
- $\mathcal{L}$ *has conjunction.*
- $\mathcal{S}_\mathcal{L} = \mathcal{B}_\mathcal{L}$ *(that is,* $\mathcal{S}_\mathcal{L}$ *is closed under non-empty finite intersections).*
- $\mathcal{S}(G) = \mathcal{B}(G)$.

Proof. We put $\sigma_\mathcal{L} := \{\cup\delta \mid \delta \subseteq \mathcal{B}_\mathcal{L})\}$. The basis of the logic $\mathcal{L}$, $\mathcal{B}_\mathcal{L}$, is by definition closed under non-empty finite intersections and $Th_\mathcal{L} = \cup\{\rho \mid \rho \in \mathcal{B}_\mathcal{L}\}$. These two conditions are sufficient for $\mathcal{B}_\mathcal{L}$ being a basis of the topology $\sigma_\mathcal{L}$ on $Th_\mathcal{L}$. It is easy to see that the set of all finite intersections of elements of $\mathcal{S}_\mathcal{L}$ (inclusive the empty intersection $\cap\varnothing = Th_\mathcal{L}$) forms also a basis of the topology $\sigma_\mathcal{L}$. Hence, $\mathcal{S}_\mathcal{L}$ is a subbasis of $\sigma_\mathcal{L}$.

Now suppose that the assumptions of part (ii) of the proposition are true. We show only the equivalence of the first and the second condition. The equivalence of the first and the third condition follows by a similar argumentation and by the fact that G is a set of generators.

Suppose $\mathcal{L}$ has conjunction. It is sufficient to show that $\mathcal{S}_\mathcal{L}$ is closed under non-empty finite intersections. By the definition of conjunction, for any two expressions a, b there is some expression c such that $c^{*_\mathcal{L}} = a^{*_\mathcal{L}} \cap b^{*_\mathcal{L}}$. Furthermore, if $\varnothing \in Th_\mathcal{L}$, then $\{\varnothing\} \in \mathcal{S}_\mathcal{L}$. The intersection of this set with any other element of the subbasis is empty. So in order to prove that the subbasis is closed under non-empty finite intersections, it remains to show that the empty set is an element of $\mathcal{S}_\mathcal{L}$ whenever $\varnothing \in Th_\mathcal{L}$. But this is guaranteed by hypothesis, $\bot^{*_\mathcal{L}} = \varnothing \in \mathcal{S}_\mathcal{L}$. It follows that $\mathcal{S}_\mathcal{L}$ is closed under non-empty finite intersections. It is evident that $Th_\mathcal{L} = \cup\mathcal{S}_\mathcal{L}$. Hence, $\mathcal{S}_\mathcal{L}$ is a basis of $\sigma_\mathcal{L}$. Moreover, we have also shown that $\mathcal{S}_\mathcal{L} = \mathcal{B}_\mathcal{L}$. On the other hand, if $\mathcal{S}_\mathcal{L}$ is closed under non-empty finite intersections, then it follows that $\mathcal{L}$ has conjunction. □

Remark 2.10. Alternatively, it is also possible to define the subbasis as $\mathcal{S}(\mathcal{T}) := \{a^\mathcal{T} \mid a \in Expr_\mathcal{L}\}$ and the basis $\mathcal{B}(\mathcal{T})$ as the set of all finite intersections of elements of $\mathcal{S}(\mathcal{T})$ (inclusive the empty intersection $\cap\varnothing = \mathcal{T}$), for any $\mathcal{T} \subseteq Th_\mathcal{L}$. If the empty set is a theory, then it is not necessary to consider this in a seperate case, as in the definition of $\mathcal{S}(\mathcal{T})$ and $\mathcal{B}(\mathcal{T})$ above. One gets a topology that is coarser than the topology defined above, where the empty theory is an isolated point, i.e., the unique element of an (basic) open set, whereas here the empty theory is an element of the open set $\mathcal{T}$ (i.e., not isolated).

Note that both topologies differ only in the treatment of the empty theory (if it exists). We choose for our work the finer topology given by the basis above.

However, it seems that the choice between these two alternatives is not really relevant for our research.

Definition 2.11. Let $\mathcal{L}$ be an abstract logic. The topological space $(Th_{\mathcal{L}}, \sigma_{\mathcal{L}})$ given by the basis $\mathcal{B}_{\mathcal{L}}$ is called the space of $\mathcal{L}$. For this space we often write $Th_{\mathcal{L}}$.

Definition 2.12. We say that a abstract logic is compact, if its space $Th_{\mathcal{L}}$ is compact.

Problem: Is a compact abstract logic finitary? Is there some finitary logic which is not compact?

We have defined two different notions of finiteness. The first one is based on the concept of consistency or - in a model-theoretic context - satisfaction. The second one is based on the concept of deduction or consequence. The following result gives a sufficient condition for the equivalence of these two notions.

Theorem 2.13. *Let $\mathcal{L}$ be an abstract logic.*
- (i) *If $\mathcal{L}$ has a classical negation $\sim$ and $\mathcal{L}$ is finitary, then $\Vdash_{\mathcal{L}}$ is finitary.*
- (ii) *If $\Vdash_{\mathcal{L}}$ is finitary and there exists some finite, inconsistent set C, then $\mathcal{L}$ is finitary.*

Proof. Suppose that $\sim$ is classical and let $\mathcal{L}$ be finitary. Let $A \Vdash_{\mathcal{L}} a$. If A is inconsistent, then there is a finite inconsistent subset A_f of A. Thus, $A_f \Vdash_{\mathcal{L}} b$ for all $b \in Expr_{\mathcal{L}}$. So we may assume that A is consistent. Towards a contradiction suppose that $A_f \nVdash_{\mathcal{L}} a$, for all finite subsets A_f. Then for each finite subset A_f there is some theory $T \supseteq A_f$ such that $a \notin T$. Since $\sim$ is a classical negation, $MTh_{\mathcal{L}}$ is the minimal generator set. Hence, $T = \cap \mathcal{T}$ for some $\mathcal{T} \subseteq MTh_{\mathcal{L}}$. Thus, $a \notin T_m$ for some $T_m \in MTh_{\mathcal{L}}$ and $A_f \subseteq T_m$. Then $\sim a \in T_m$. Hence, $A_f \cup \{\sim a\}$ is consistent, for all finite $A_f \subseteq A$. Since $\mathcal{L}$ is finitary, $A \cup \{\sim a\}$ is consistent. It follows: $A \nVdash_{\mathcal{L}} a$, a contradiction to our hypothesis. Thus, (i) is true.

Now suppose that $\Vdash_{\mathcal{L}}$ is finitary and C is a finite, inconsistent set of expressions. Then, in particular, $Expr_{\mathcal{L}} \notin Th_{\mathcal{L}}$, i.e., $\mathcal{L}$ is regular. Let $A \subseteq Expr_{\mathcal{L}}$ such that every finite subset of A is consistent. Towards a contradiction we assume that A is inconsistent. It follows that $A \Vdash_{\mathcal{L}} Expr_{\mathcal{L}}$. In particular, $A \Vdash_{\mathcal{L}} C$, that is, $A \Vdash_{\mathcal{L}} c$ for each $c \in C$. Since $\Vdash_{\mathcal{L}}$ is finitary, for each $c \in C$ there is some finite subset $A_c \subseteq A$ such that $A_c \Vdash_{\mathcal{L}} c$. By monotonicity of the consequence relation, $B \Vdash_{\mathcal{L}} C$, where $B := \cup\{A_c \mid c \in C\}$. Thus, B is inconsistent. But B is a finite subset of A, a contradiction. Hence, A is consistent and the assertion (ii) holds. □

Corollary 2.14. *If $\mathcal{L}$ is a logic with classical negation, then $\mathcal{L}$ is finitary if and only if $\Vdash_{\mathcal{L}}$ is finitary.*

In order to give an example that points out the connection between concepts of our abstract logics and topological terms we prove the following result. It says that each boolean abstract logic gives rise to a boolean topological space. It also shows that in the case of boolean abstract logics the notions “finitary” and “compact” coincide.

Theorem 2.15. *Let $\mathcal{L}$ be an abstract logic. Suppose that the set of maximal theories $G := MTh_{\mathcal{L}}$ is a set of generators of the logic $\mathcal{L}$. Then holds the following: $\mathcal{L}$ is a boolean logic if and only if $\mathcal{S}(G) = \mathcal{B}(G)$ is closed under finite intersections and under complement*[7] *and is a basis of a boolean space on $G = MTh_{\mathcal{L}}$ (i.e., a topological space, which is compact, Hausdorff and has a basis of clopen sets).*

Proof. Suppose that $\mathcal{L}$ is a boolean logic. Then clearly $\mathcal{S}(G) = \mathcal{B}(G)$. Let $a^G \in \mathcal{B}(G)$ and let b be a classical negation of a. Then follows that $b^G = MTh_{\mathcal{L}} \smallsetminus a^G$, hence, the elements of the basis are clopen. Moreover, this also shows that the basis is closed under complement. Since the logic has conjunction the basis is also closed under finite intersections. In order to see that the topology is Hausdorff let $T_1, T_2 \in G$ such that $T_1 \neq T_2$. We may assume that there is some $a \in T_1 \smallsetminus T_2$. Let b be a classical negation of a. Then $T_1 \in a^G$, $T_2 \in b^G$ and $a^G \cap b^G = \varnothing$.

Finally, we show that the topology given by the basis $\mathcal{B}(G)$ is compact. So assume that $G = MTh_{\mathcal{L}} = \cup_{i\in I} U_i$ for some system of open sets $U_i = \cup_{a\in A_i} a^G$, $A_i \subseteq Expr_{\mathcal{L}}$. Put $A := \cup_{i\in I} A_i$. Then

$$\begin{aligned}\varnothing &= MTh_{\mathcal{L}} \smallsetminus \cup\{U_i \mid i \in I\} \\ &= MTh_{\mathcal{L}} \smallsetminus \cup\{a^G \mid a \in A\} \\ &= \cap\{MTh_{\mathcal{L}} \smallsetminus a^G \mid a \in A\} \\ &= \cap\{b^G \mid b \text{ is negation of some } a \in A\}\,.\end{aligned}$$

(Note that two classical negations of the same expression a are $\mathcal{L}$-equivalent.)

That is, the set $B := \{b \mid b \text{ is negation of some } a \in A\}$ is not contained in any maximal theory. Since every theory is contained in some maximal theory, it follows that B is not contained in any theory, that is, B is inconsistent. Since $\mathcal{L}$ is finitary, there is some finite subset $B_f \subseteq B$ such that B_f is inconsistent. It follows that there is some finite subset $A_f \subseteq A$ such that $B_f = \{b \mid b \text{ is negation of some } a \in A_f\}$. Hence, $\cap\{b^G \mid b \text{ is negation of some } a \in A_f\} = \varnothing$. Then we get the following:

$$\begin{aligned}G &= G \smallsetminus \cap\{b^G \mid b \text{ is negation of some } a \in A_f\} \\ &= \cup\{G \smallsetminus b^G \mid b \text{ is negation of some } a \in A_f\} \\ &= \cup\{a^G \mid a \in A_f\}\,.\end{aligned}$$

Thus, there is also a finite subset $I_f \subseteq I$ such that $G = \cup_{i\in I_f} U_i$. Hence, the topology given by the basis $\mathcal{B}(G)$ is compact.

Now suppose that $\mathcal{B}(G) = \mathcal{S}(G)$ forms a basis of a boolean space and that this basis is closed under finite intersections and under complement. By Proposition 2.9, $\mathcal{L}$ has conjunction. Furthermore, for any expression a, $G \smallsetminus a^G \in \mathcal{B}(G)$, since the basis is closed under complement. It follows that the logic has classical negation.

It remains to show that $\mathcal{L}$ is finitary. So let $A \subseteq Expr_{\mathcal{L}}$ be inconsistent. We show the existence of a finite and inconsistent subset of A.

[7]If X is a family of sets, then we say that X is closed under complement, if for every $Y \in X$, $\cup X \smallsetminus Y \in X$.

Since A is inconsistent, $\cap\{a^{*_\mathcal{L}} \mid a \in A\} = \varnothing$. In particular, $\cap\{a^G \mid a \in A\} = \varnothing$. (Note that $a^G = a^{*_\mathcal{L}} \cap G$.) Therefore,

$$\begin{aligned} G = MTh_\mathcal{L} &= MTh_\mathcal{L} \smallsetminus \cap\big\{a^G \mid a \in A\big\} \\ &= \cup\big\{Th_\mathcal{L} \smallsetminus a^G \mid a \in A\big\} \\ &= \cup\big\{b^G \mid b \text{ is a negation of some } a \in A\big\}\,. \end{aligned}$$

Since the topology is compact, there is a finite subset $A_f \subseteq A$ such that $G = MTh_\mathcal{L} = \cup\{b^G \mid b \text{ is a negation of some } a \in A_f\}$. We get

$$\begin{aligned} \varnothing &= G \smallsetminus \cup\big\{b^G \mid b \text{ is a negation of some } a \in A_f\big\} \\ &= \cap\big\{G \smallsetminus b^G \mid b \text{ is a negation of some } a \in A_f\big\} \\ &= \cap\big\{a^G \mid a \in A_f\big\}\,, \end{aligned}$$

that is, $A_f \subseteq A$ is inconsistent. Thus, $\mathcal{L}$ is finitary. □

Remark 2.16. Let $\mathcal{L}$ be a classical abstract logic. The open sets of the subspace topology on $MTh_\mathcal{L}$ with basis $\mathcal{B}(MTh_\mathcal{L}) = \mathcal{S}(MTh_\mathcal{L})$ are exactly the sets $\cup\{a^{MTh_\mathcal{L}} \mid a \in A\}$, for $A \subseteq Expr_\mathcal{L}$. The closed sets of this subspace topology are exactly the sets $\cap\{a^{MTh_\mathcal{L}} \mid a \in A\} = \{T \in MTh_\mathcal{L} \mid A \subseteq T\} = A^{MTh_\mathcal{L}}$, for $A \subseteq Expr_\mathcal{L}$. Hence, there exists a bijection

$$A^{MTh_\mathcal{L}} \mapsto \cap\{T \in MTh_\mathcal{L} \mid A \subseteq T\}\,, \quad \text{for} \quad A \subseteq Expr_\mathcal{L}\,,$$

between the closed sets of the subspace topology on $MTh_\mathcal{L}$ and the set of all deductively closed sets of the logic $\mathcal{L}$ (notice that the deductively closed sets are exactly all theories together with $Expr_\mathcal{L}$).

3. Logic maps

Definition 3.1. Suppose that $\mathcal{L} = (Expr_\mathcal{L}, Th_\mathcal{L})$ and $\mathcal{L}' = (Expr_{\mathcal{L}'}, Th_{\mathcal{L}'})$ are abstract logics. Let $h : Expr_\mathcal{L} \to Expr_{\mathcal{L}'}$ be a function. We say that h is logically injective, if $a \neq_\mathcal{L} b$ implies $h(a) \neq_{\mathcal{L}'} h(b)$. h is logically surjective, if for every $a' \in Expr_{\mathcal{L}'}$ there is some $a \in Expr_\mathcal{L}$ such that $h(a) =_{\mathcal{L}'} a'$. We write L-injective (L-surjective) for logically injective (logically surjective), respectively.[8] h is regular, if $a =_\mathcal{L} b$ implies $h(a) =_{\mathcal{L}'} h(b)$, for all $a, b \in Expr_\mathcal{L}$.

Finally, h is a logic map from $\mathcal{L}$ to $\mathcal{L}'$, if the following holds:

$$\big\{h^{-1}(T') \mid T' \in Th_{\mathcal{L}'}\big\} \subseteq Th_\mathcal{L}\,. \tag{3.1}$$

We write $h : \mathcal{L} \to \mathcal{L}'$, if h is a logic map from the logic $\mathcal{L}$ to the logic $\mathcal{L}'$.

We collect some basic properties of logic maps.

Proposition 3.2. *Let $h : \mathcal{L} \to \mathcal{L}'$ be a logic map. Then for all $A \subseteq Expr_\mathcal{L}$ and all $a, b \in Expr_\mathcal{L}$ it holds the following:*

[8]Note that injectivity of h does not imply L-injectivity of h, neither L-injectivity implies injectivity of h. On the other hand, surjectivity of h clearly implies L-surjectivity of h.

(i) *If* $A' \subseteq Expr_{\mathcal{L}'}$ *is consistent in* $\mathcal{L}'$*, then* $h^{-1}(A')$ *is consistent in* $\mathcal{L}$.
(ii) *If* $A \Vdash_{\mathcal{L}} B$*, then* $h(A) \Vdash_{\mathcal{L}'} h(B)$*, for all* $A, B \subseteq Expr_{\mathcal{L}}$.
(iii) *If* $a =_{\mathcal{L}} b$*, then* $h(a) =_{\mathcal{L}'} h(b)$*, that is,* h *is regular.*
(iv) $h^{-1}(h(A)^{*_{\mathcal{L}'}}) \subseteq A^{*_{\mathcal{L}}}$ *and* $h^{-1}(h(a)^{*_{\mathcal{L}'}}) \subseteq a^{*_{\mathcal{L}}}$.
(v) *If* $\mathcal{L}'$ *is singular, then* $\mathcal{L}$ *is singular.*

Proof. (i) If $A' \subseteq Expr_{\mathcal{L}'}$ is consistent in $\mathcal{L}'$, then it must be contained in some theory T'. From the definition it follows that $h^{-1}(A')$ is contained in a theory of $\mathcal{L}$, thus it is consistent in $\mathcal{L}$.

(ii) If $h(A)$ is inconsistent, then the assertion is trivial. So let us suppose that $A \Vdash_{\mathcal{L}} B$ and $h(A)$ is consistent. Let T' be any theory in $\mathcal{L}'$ such that $h(A) \subseteq T'$. Then $A \subseteq h^{-1}(h(A)) \subseteq h^{-1}(T') = T$ for some $T \in Th_{\mathcal{L}}$. Since $A \Vdash_{\mathcal{L}} B$, it follows that $B \subseteq T = h^{-1}(T')$, thus $h(B) \subseteq T'$. Since T' was arbitrary, it follows that $h(B) \subseteq \cap\{T' \in Th_{\mathcal{L}'} \mid h(A) \subseteq T'\}$.

(iii) $a =_{\mathcal{L}} b$ is defined by $a \Vdash_{\mathcal{L}} b$ and $b \Vdash_{\mathcal{L}} a$. Now the assertion follows immediately from (ii).

(iv) Let $T \in h^{-1}(h(A)^{*_{\mathcal{L}'}})$. Then there is some $T' \in h(A)^{*_{\mathcal{L}'}}$ such that $T = h^{-1}(T')$. Since $h(A) \subseteq T'$, $A \subseteq T$. That is, $T \in A^{*_{\mathcal{L}}}$. The second assertion follows analogously.

(v) Let $\mathcal{L}'$ be a singular logic, that is, $Expr_{\mathcal{L}'} \in Th_{\mathcal{L}'}$. Since $h^{-1}(Expr_{\mathcal{L}'}) = Expr_{\mathcal{L}}$ and h is a logic map, $Expr_{\mathcal{L}} \in Th_{\mathcal{L}}$ and $\mathcal{L}$ is singular. □

In the literature, a logic translation (or translation of logics) is usually defined as a map from the language (formulas) of one logic to the language of another logic preserving the consequence relation. Sometimes one requires some additional conditions. For instance, a uniform translation satisfies some syntactical rules (it is an extension of a morphism between signatures). Many translations relevant in practice are not uniform. For a discussion and a historical overview on this subject see [3]. See also [2], where a uniform translation is defined in order to develop a concept of equivalence between logics.

In the context of abstract logics we may define a more relaxed notion of translation as follows: A translation from $\mathcal{L}$ to $\mathcal{L}'$ is a function $g : Expr_{\mathcal{L}} \rightarrow Expr_{\mathcal{L}'}$ such that $A \Vdash_{\mathcal{L}} a$ implies $g(A) \Vdash_{\mathcal{L}'} g(a)$, for all $A \cup \{a\} \subseteq Expr_{\mathcal{L}}$.

We have seen in the previous proposition that a logic map is a translation in this general sense. The following result gives a sufficient condition for the equivalence of logic maps and translations between abstract logics:

Proposition 3.3. *Let* $\mathcal{L}, \mathcal{L}'$ *be abstract logics and let* $h : Expr_{\mathcal{L}} \rightarrow Expr_{\mathcal{L}'}$ *be a function. Suppose that the following holds:* $\mathcal{L}$ *is singular or there are inconsistent sets* $C \subseteq Expr_{\mathcal{L}}$ *and* $C' \subseteq Expr_{\mathcal{L}'}$ *such that* $h(C) = C'$*. Then the following conditions are equivalent:*

(i) h *is a logic map.*
(ii) *If* $A \Vdash_{\mathcal{L}} a$*, then* $h(A) \Vdash_{\mathcal{L}'} h(a)$*, for all* $A \cup \{a\} \subseteq Expr_{\mathcal{L}}$.

Proof. The implication (i)→(ii) we have already proved in the preceding proposition. Suppose that (ii) holds. Let $T' \in Th_{\mathcal{L}'}$ and $h^{-1}(T') = T$. In order to prove that T is an $\mathcal{L}$-theory we show that T is deductively closed and consistent. So let us assume that $T \Vdash_{\mathcal{L}} a$, for some $\mathcal{L}$-expression a. Then $h(T) \Vdash_{\mathcal{L}'} h(a)$, by hypothesis. Since $h(T) \subseteq T'$, $T' \Vdash_{\mathcal{L}'} h(a)$. Since T' is deductively closed, we get $h(a) \in T'$, thus $a \in T$. Hence, T is deductively closed.

If $\mathcal{L}$ is singular, then every set is consistent. Thus, T is consistent and therefore a theory. Let us assume that $\mathcal{L}$ is regular. Towards a contradiction we suppose that T is inconsistent. Then every set of $\mathcal{L}$-expressions, in particular C, is a consequence of T. By hypothesis, $h(T) \Vdash_{\mathcal{L}'} h(C)$. By monotonicity, $T' \Vdash_{\mathcal{L}'} C'$. It follows that T' is inconsistent, a contradiction. Hence, T is consistent. □

We use this result to show that the Gödel translation from classical to intuitionistic propositional logic is a logic map:

Example 4. Suppose that $\mathcal{L}$ and $\mathcal{L}'$ are the classical and intuitionistic propositional logic, respectively, as defined in Examples 2 and 3. We define inductively a function $g : Expr_{\mathcal{L}} \to Expr_{\mathcal{L}'}$ in the following way:

$$\begin{aligned} g(p) &:= \sim\sim p, \text{ for atomic } p \in P \\ g(\sim a) &:= \sim g(a) \\ g(a \wedge b) &:= g(a) \wedge g(b) \\ g(a \vee b) &:= \sim \big(\sim g(a) \wedge \sim g(b) \big) \\ g(a \to b) &:= g(a) \to g(b)\,. \end{aligned}$$

The function g is known in the literature as the Gödel translation. It is well-known that $A \Vdash_{\mathcal{L}} a$ implies $g(A) \Vdash_{\mathcal{L}'} g(a)$ (and vice-versa), for all $A \cup \{a\} \subseteq Expr_{\mathcal{L}}$. Furthermore, for any $p \in P$, the expressions $a := p \wedge \sim p$ and $g(a) = g(p) \wedge \sim g(p) = \sim\sim p \wedge \sim\sim\sim p$ are inconsistent in the respective logics. Now we can apply Proposition 3.3, which says that g is a logic map.

An important case of logic maps is given if we can substitute inclusion by equality in the defining condition.

Definition 3.4. A logic map $h : \mathcal{L} \to \mathcal{L}'$ is called normal, if $\{h^{-1}(T') \mid T' \in Th_{\mathcal{L}'}\} = Th_{\mathcal{L}}$.

Lemma 3.5. *A logic map $h : \mathcal{L} \to \mathcal{L}'$ is normal if and only if $h^{-1}(h(a)^{*_{\mathcal{L}'}}) = a^{*_{\mathcal{L}}}$, for all $a \in Expr_{\mathcal{L}}$, if and only if $h^{-1}(h(A)^{*_{\mathcal{L}'}}) = A^{*_{\mathcal{L}}}$, for all $A \subseteq Expr_{\mathcal{L}}$.*

Proof. Let h be normal. Since h is a logic map, Proposition 3.2 yields the inclusion $h^{-1}(h(a)^{*_{\mathcal{L}'}}) \subseteq a^{*_{\mathcal{L}}}$. Let $T \in a^{*_{\mathcal{L}}}$. There is some $T' \in Th_{\mathcal{L}'}$ such that $h^{-1}(T') = T$. Then $h(a) \in T'$ and $T' \in h(a)^{*_{\mathcal{L}'}}$. Hence, $T \in h^{-1}(h(a)^{*_{\mathcal{L}'}})$. Thus, the assertion follows.

Now suppose that $h^{-1}(h(a)^{*_{\mathcal{L}'}}) = a^{*_{\mathcal{L}}}$ holds, for all $a \in Expr_{\mathcal{L}}$. Let $T \in Th_{\mathcal{L}}$. We must show that there is some $T' \in Th_{\mathcal{L}'}$ such that $h^{-1}(T') = T$. First, we suppose that $T \neq \varnothing$. Then there is some $a \in T$, that is, $T \in a^{*_{\mathcal{L}}}$. Now, the existence of a $T' \in Th_{\mathcal{L}'}$ such that $h^{-1}(T') = T$ follows readily from the hypothesis. Finally, suppose that T is the empty theory. Then there is some ordinal α and theories T_i such that $T \neq T_i$, $i < \alpha$, and $T = \cap\{T_i \mid i < \alpha\}$. As we have already seen, for each T_i there is some T_i' such that $h^{-1}(T_i') = T_i$. Then $T = \cap\{T_i \mid i < \alpha\} = \cap\{h^{-1}(T_i') \mid i < \alpha\} = h^{-1}(\cap\{T_i' \mid i < \alpha\}) = h^{-1}(T')$, where $T' = \cap\{T_i' \mid i < \alpha\} \in Th_{\mathcal{L}'}$.

The proof works in the same way, if we assume a set $A \subseteq Expr_{\mathcal{L}}$ instead of a single expression. □

Lemma 3.6. *If $h : \mathcal{L} \to \mathcal{L}'$ is normal, then h is L-injective.*

Proof. Suppose that h is normal. Let $a, b \in Expr_{\mathcal{L}}$ such that $h(a) =_{\mathcal{L}'} h(b)$. This is equivalent to the condition $h(a)^{*_{\mathcal{L}'}} = h(b)^{*_{\mathcal{L}'}}$. By Lemma 3.5, $a^{*_{\mathcal{L}}} = h^{-1}(h(a)^{*_{\mathcal{L}'}}) = h^{-1}(h(b)^{*_{\mathcal{L}'}}) = b^{*_{\mathcal{L}}}$. That is, $a =_{\mathcal{L}} b$. □

We outline an example of a logic map which is not normal:

Example 5. Suppose that $\mathcal{L}_I$ is the intuitionistic propositional logic and $\mathcal{L}_{Cl}$ is the classical propositional logic as defined in Examples 2 and 3, respectively. Note that $Expr_{\mathcal{L}_I} = Expr_{\mathcal{L}_{Cl}}$. The set of maximal $\mathcal{L}_{Cl}$-theories is the smallest generator set for $\mathcal{L}_{Cl}$. Each maximal (complete) $\mathcal{L}_{Cl}$-theory is also an $\mathcal{L}_I$-theory. Therefore, the identity map $i : Expr_{\mathcal{L}_I} \to Expr_{\mathcal{L}_{Cl}}$, $a \mapsto a$, is a logic map: $i^{-1}(T) = \{a \in Expr_{\mathcal{L}_I} \mid i(a) = a \in T\} = T \in Th_{\mathcal{L}_I}$, for all $T \in Th_{\mathcal{L}_{Cl}}$. It is well-known that for an arbitrary expression a, $a =_{\mathcal{L}_{Cl}} \sim\sim a$ but $a \neq_{\mathcal{L}_I} \sim\sim a$. Hence, i is not L-injective. By the previous Lemma, i can not be normal.

Here comes an example of a normal logic map:

Example 6. We consider first order logic in some given signature (language) Σ. First order logic has an extension[9] in which exist formulas for conjunctions and disjunctions of (countable) infinite sets of formulas. (For instance, if Ω is an countable infinite set of formulas, then $\bigwedge \Omega$ is a formula.) This logic is usually denoted by $\mathcal{L}_{\omega_1\omega}$. Let $Expr_{\mathcal{L}}$ be the set of first order sentences over Σ and let $Expr_{\mathcal{L}'}$ be the set of $\mathcal{L}_{\omega_1\omega}$-sentences over Σ. As interpretations we take the class of all Σ-structures. The respective satisfaction relations $\vDash_{\mathcal{L}}$ and $\vDash_{\mathcal{L}'}$ are defined in the usual way. Then $\mathcal{L}$ and $\mathcal{L}'$ are the respective abstract logics generated by the class of all Σ-structures. The point is that both abstract logics are generated by the same class of models. Each maximal theory $T' \in MTh_{\mathcal{L}'}$ is the complete theory of some model, say $\mathcal{A}$. Then $T' \cap Expr_{\mathcal{L}} = T$ is a theory, namely the complete (=maximal) theory in $\mathcal{L}$ of the same model $\mathcal{A}$. Conversely, for each $T \in MTh_{\mathcal{L}}$ there is exactly one $T' \in Expr_{\mathcal{L}'}$ such that $T = T' \cap Expr_{\mathcal{L}}$ and T, T' are complete theories of the same model, in their respective logics. This relationship yields a bijection between $MTh_{\mathcal{L}}$ and $MTh_{\mathcal{L}'}$. Now it follows easily that there is also a bijection between $Th_{\mathcal{L}}$ and $Th_{\mathcal{L}'}$. Let $T \mapsto T'$ be this bijection. Let $i : Expr_{\mathcal{L}} \to Expr_{\mathcal{L}'}$, $a \mapsto a$, be the identity function (which is an embedding but not L-surjective). Then we get $i^{-1}(T') = \{a \in Expr_{\mathcal{L}} \mid i(a) = a \in T'\} = T' \cap Expr_{\mathcal{L}} = T \in Th_{\mathcal{L}}$. That is, i is a normal logic map.

[9]Later on we will give a precise definition of the concept of extension, see Definition 4.11.

Although i is a normal logic map (and a translation) from $\mathcal{L}$ to $\mathcal{L}'$, we will see later that i is not a logic homomorphism.

Definition 3.7. Suppose that $\mathcal{L}$ and $\mathcal{L}'$ are abstract logics. Let $h : Expr_{\mathcal{L}} \to Expr_{\mathcal{L}'}$ be a function. We say that a function $H : Th_{\mathcal{L}} \to Th_{\mathcal{L}'}$ is a complement of h, if $h^{-1}(H(T)) = T$, for all $T \in Th_{\mathcal{L}}$.

Theorem 3.8. *Suppose that $\mathcal{L}$ and $\mathcal{L}'$ are abstract logics and let $h : Expr_{\mathcal{L}} \to Expr_{\mathcal{L}'}$ be a logic map. Then h is normal if and only if h has a complement. Every complement of h is injective.*

Proof. First we suppose that h is normal. We have $\{h^{-1}(T') \mid T' \in Th_{\mathcal{L}'}\} = Th_{\mathcal{L}}$. Then for any theory T of $\mathcal{L}$ there is some theory T' of $\mathcal{L}'$ such that $h^{-1}(T') = T$. Such a T' is not necessarily unique. We may use the axiom of choice in order to define a complement H of h. Alternatively, we may define $H : Th_{\mathcal{L}} \to Th_{\mathcal{L}'}$ by $T \mapsto \cap\{T' \in Th_{\mathcal{L}'} \mid h^{-1}(T') = T\}$. Then H is a complement of h.

On the other hand, if H is a complement of h, then it is clear by the definition that h is normal.

In order to prove that any complement H of h is injective suppose that $T_1 \neq T_2$ are two $\mathcal{L}$-theories. If H maps T_1 to T'_1 and T_2 to T'_2, then

$$h^{-1}(T'_1) = T_1 \neq T_2 = h^{-1}(T'_2)\,.$$

It follows that $T'_1 \neq T'_2$ and H is injective. □

Remark 3.9. Let $h : \mathcal{L} \to \mathcal{L}'$ be a normal logic map from a logic $\mathcal{L}$ to a logic $\mathcal{L}'$. Consider the function $H : Th_{\mathcal{L}} \to Th_{\mathcal{L}'}$ defined by $T \mapsto \cap\{T' \in Th_{\mathcal{L}'} \mid h^{-1}(T') = T\}$. It is clear that H is a complement of h. Furthermore, it is easy to see that $H(T)$ is the deductive closure of $h(T)$ in $\mathcal{L}'$.

Thus, the map $T \mapsto h(T)^{\Vdash_{\mathcal{L}'}}$ is always a complement of $h : \mathcal{L} \to \mathcal{L}'$, if h is a normal logic map. We may call this complement the closure complement of h. Though, this is not always the complement we are looking for. In order to preserve the topological structure of the logic we wish that a complement H of h maps a maximal (or prime) theory of $\mathcal{L}$ to a maximal (prime) theory of $\mathcal{L}'$. We leave it as a claim here that this is, in general, not the case for the closure complement.

If $h : Expr_{\mathcal{L}} \to Expr_{\mathcal{L}'}$ is a normal logic map, then for each $T \in Th_{\mathcal{L}}$ there is some $T' \in Th_{\mathcal{L}'}$ such that $h^{-1}(T') = T$. A special case is given if for each $T \in Th_{\mathcal{L}}$ there exists exactly one $T' \in Th_{\mathcal{L}'}$ with this property:

Definition 3.10. We say that a logic map $h : Expr_{\mathcal{L}} \to Expr_{\mathcal{L}'}$ is stationary, if $h^{-1}(T'_1) = h^{-1}(T'_2)$ implies that $T'_1 = T'_2$, for any $T'_1, T'_2 \in Th_{\mathcal{L}'}$.

Example 7. Assume the special case $Expr_{\mathcal{L}} \subseteq Expr_{\mathcal{L}'}$ and the identity map $i : Expr_{\mathcal{L}} \to Expr_{\mathcal{L}'}$, $a \mapsto a$, is a normal, stationary logic map from $\mathcal{L} = (Expr_{\mathcal{L}}, Th_{\mathcal{L}})$ to $\mathcal{L}' = (Expr_{\mathcal{L}'}, Th_{\mathcal{L}'})$. Then each theory $T \in Th_{\mathcal{L}}$ has exactly one extension in logic $\mathcal{L}'$ to a theory $T' \in Th_{\mathcal{L}'}$. (Note that $i^{-1}(T') = T$ iff $T = T' \cap Expr_{\mathcal{L}}$.) Such a logic map is given in Example 6.

Proposition 3.11. *A normal and stationary logic map has exactly one comlement. The complement is bijective.*

Proof. Let $h : Expr_{\mathcal{L}} \to Expr_{\mathcal{L}'}$ be a normal and stationary logic map from $\mathcal{L}$ to $\mathcal{L}'$. By Theorem 3.8, h has a complement H. Let H' be also a complement of h. Then for all $T \in Th_{\mathcal{L}}$, $h^{-1}(H(T)) = T = h^{-1}(H'(T))$. Since h is stationary, $H(T) = H'(T)$, for all $\mathcal{L}$-theories T. Hence, $H = H'$.

In order to show that the complement H is bijective, suppose that $T' \in Th_{\mathcal{L}'}$. Then $h^{-1}(T') = T \in Th_{\mathcal{L}}$. Let $H(T) = T''$. Since $h^{-1}(T'') = h^{-1}(H(T)) = T = h^{-1}(T')$, it follows that $T' = T''$. Hence, $H(T) = T'$ and H is surjective. H is injective by Theorem 3.8. □

Proposition 3.12. *Suppose that $\mathcal{L}$ and $\mathcal{L}'$ are abstract logics. Let $h : Expr_{\mathcal{L}} \to Expr_{\mathcal{L}'}$ and $H : Th_{\mathcal{L}} \to Th_{\mathcal{L}'}$ be functions. Then the following conditions are equivalent:*

(i) *H is a complement of h.*
(ii) *$H^{-1}(h(a)^{*_{\mathcal{L}'}}) = a^{*_{\mathcal{L}}}$, for all $a \in Expr_{\mathcal{L}}$.*
(iii) *$a \in T \iff h(a) \in H(T)$, for all $a \in Expr_{\mathcal{L}}$ and all $T \in Th_{\mathcal{L}}$.*
(iv) *$T \vDash_{\mathcal{L}} a \iff H(T) \vDash_{\mathcal{L}'} h(a)$, for all $a \in Expr_{\mathcal{L}}$ and all $T \in Th_{\mathcal{L}}$.*

Proof. Suppose that (i) holds and let $a \in Expr_{\mathcal{L}}$. Let $T \in H^{-1}(h(a)^{*_{\mathcal{L}'}})$. Then $H(T) \in h(a)^{*_{\mathcal{L}'}}$, thus $h(a) \in H(T)$. It follows that $a \in h^{-1}(H(T)) = T$, hence $T \in a^{*_{\mathcal{L}}}$. Now suppose $T \in a^{*_{\mathcal{L}}}$. Then $a \in T$ and $h(a) \in h(T) \subseteq H(T)$, since $h^{-1}(H(T)) = T$. Thus, $H(T) \in h(a)^{*_{\mathcal{L}'}}$ and $T \in H^{-1}(h(a)^{*_{\mathcal{L}'}})$. We have proved $H^{-1}(h(a)^{*_{\mathcal{L}'}}) = a^{*_{\mathcal{L}}}$.

The proof of the implications (ii)→(iii)→(iv) is straightforeward. Notice that (iii) and (iv) are, by definition, in fact the same condition (we interprete in (iv) theories as abstract models, see Remark 2.3).

Finally, assume (iv). (iv) is by definition equivalent with (iii). We show that (iii) implies (i): Let $T \in Th_{\mathcal{L}}$. By (iii) we have $a \in h^{-1}(H(T))$ iff $h(a) \in H(T)$ iff $a \in T$. That is, (i) holds. □

Remark 3.13. Let $h : Expr_{\mathcal{L}} \to Expr_{\mathcal{L}'}$ be a normal logic map and H a complement of h. Suppose that $\varnothing \in Th_{\mathcal{L}}$. It seems that the preceding results do not imply in general that $H(\varnothing) = \varnothing$. However, it follows that there is no $a \in Expr_{\mathcal{L}}$ such that $h(a) \in H(\varnothing) \in Th_{\mathcal{L}'}$. For otherwise there would be an $a \in Expr_{\mathcal{L}}$ such that $H(\varnothing) \in h(a)^{*_{\mathcal{L}'}}$. By (ii) of the preceding Proposition, $\varnothing \in H^{-1}(h(a)^{*_{\mathcal{L}'}}) = a^{*_{\mathcal{L}}}$, that is, $a \in \varnothing$, a contradiction.

Proposition 3.14. *Let $h : Expr_{\mathcal{L}} \to Expr_{\mathcal{L}'}$ be a normal logic map and H a complement of h.*

(i) *$H^{-1}(h(a)^{*_{\mathcal{L}'}}) = a^{*_{\mathcal{L}}} = h^{-1}(h(a)^{*_{\mathcal{L}'}})$, for all $a \in Expr_{\mathcal{L}}$.*
(ii) *$H(a^{*_{\mathcal{L}}}) \subseteq h(a)^{*_{\mathcal{L}'}}$, for all $a \in Expr_{\mathcal{L}}$. If h is stationary, then $H(a^{*_{\mathcal{L}}}) = h(a)^{*_{\mathcal{L}'}}$, for all $a \in Expr_{\mathcal{L}}$.*
(iii) *$H^{-1}(a'^{*_{\mathcal{L}'}}) \subseteq h^{-1}(a'^{*_{\mathcal{L}'}})$, for all $a' \in Expr_{\mathcal{L}'}$. If h is stationary, then $H^{-1}(a'^{*_{\mathcal{L}'}}) = h^{-1}(a'^{*_{\mathcal{L}'}})$, for all $a' \in Expr_{\mathcal{L}'}$.*

Proof. (i) This follows from Proposition 3.12 and Lemma 3.5.

(ii) $H(a^{*_{\mathcal{L}'}}) \subseteq h(a)^{*_{\mathcal{L}'}}$ follows from (ii) of the preceding proposition. Now let h be stationary and let $T' \in h(a)^{*_{\mathcal{L}'}}$. Then $h^{-1}(T') = T$, for some $T \in a^{*_{\mathcal{L}}}$. Suppose that $H(T) = T''$. Then $h^{-1}(T') = T = h^{-1}(T'')$. Since h is stationary, T' and T'' must be equal. Thus, $H(T) = T'$ and the assertion follows.

(iii) The first assertion is easy to prove. Now let h be stationary. It remains to prove the inclusion "$\supseteq$". Let $T \in h^{-1}(a'^{*_{\mathcal{L}'}})$. Then there is some $T' \in a'^{*_{\mathcal{L}'}}$ such that $h^{-1}(T') = T$. Since h is stationary, it follows that $H(T) = T'$. Hence, $T \in H^{-1}(a'^{*_{\mathcal{L}'}})$. □

Corollary 3.15. *Let h be a normal logic map from $\mathcal{L}$ to $\mathcal{L}'$ and let H be a complement of h. Then holds the following:*

(i) *If h is L-surjective, then H is a continuous map from the space of $\mathcal{L}$ to the space of $\mathcal{L}'$.*
(ii) *If h is stationary, then H is an open map from the space of $\mathcal{L}$ to the space of $\mathcal{L}'$.*

Proof. (i) We must show that H^{-1} maps basic opens of $\mathcal{L}'$ to open sets of $\mathcal{L}$. The basic opens of $\mathcal{L}'$ are given by $a_1'^{*_{\mathcal{L}'}} \cap \cdots \cap a_n'^{*_{\mathcal{L}'}}$, for sequences $a_1', \ldots, a_n' \in Expr_{\mathcal{L}'}$, $n \geq 1$, (and $\{\varnothing\}$, if the empty set is a theory). It is easy to see that $H^{-1}(a_1'^{*_{\mathcal{L}'}} \cap \cdots \cap a_n'^{*_{\mathcal{L}'}}) = H^{-1}(a_1'^{*_{\mathcal{L}'}}) \cap \cdots \cap H^{-1}(a_n'^{*_{\mathcal{L}'}})$ holds, for any such a sequence. It follows that in order to prove that H is continuous it suffices to show that

- $H^{-1}(a'^{*_{\mathcal{L}'}})$ is an open set, for each $a' \in Expr_{\mathcal{L}'}$, and
- if $\varnothing \in Th_{\mathcal{L}'}$, then $H^{-1}(\{\varnothing\})$ is an open set.

The first item follows immediately from (i) of the preceding proposition. Suppose that $\varnothing \in Th_{\mathcal{L}'}$. Then $h^{-1}(\varnothing) = \varnothing \in Th_{\mathcal{L}}$, since h is a logic map. By the preceding remark, there is no $a \in Expr_{\mathcal{L}}$ such that $h(a) \in H(\varnothing) \in Th_{\mathcal{L}'}$. Since h is L-surjective, it follows that $H(\varnothing) = \varnothing$. That is, $H^{-1}(\{\varnothing\}) = \{\varnothing\}$, which is an open set in the space of $\mathcal{L}$.

(ii) By Theorem 3.8, H is an injective function. It follows that $H(a^{*_{\mathcal{L}}} \cap b^{*_{\mathcal{L}}}) = H(a^{*_{\mathcal{L}}}) \cap H(b^{*_{\mathcal{L}}})$, for any $a, b \in Expr_{\mathcal{L}}$. Therefore, in order to show that H is an open map, it suffices to prove that $H(a^{*_{\mathcal{L}}})$ is an open, for each $a \in Expr_{\mathcal{L}}$. This follows from (ii) of the preceding proposition. □

Remark 3.16. We will see later (by giving counter examples) that H (in the above corollary) is in general not a continuous function, if h is a normal stationary logic map but not L-surjective. However, H will be a homeomorphism, if h is a normal stationary logic homomorphism (see Proposition 4.6), even if not L-surjective.

Corollary 3.17. *If $h : \mathcal{L} \to \mathcal{L}'$ is a normal and stationary logic map, then the (unique) complement H of h is a bijective open function from the space of $\mathcal{L}$ to the space of $\mathcal{L}'$.*

Lemma 3.18. *If $h : \mathcal{L} \to \mathcal{L}'$ is an L-surjective logic map, then h is stationary.*

Proof. Let $T', T'' \in Th_{\mathcal{L}'}$ such that $h^{-1}(T') = h^{-1}(T'') := T \in Th_{\mathcal{L}}$. Then $h(T) \subseteq T', T''$. Since h is L-surjective, for each $a' \in T'$ there is some $a \in Expr_{\mathcal{L}}$ such that $h(a) =_{\mathcal{L}'} a'$. T' is deductively closed and it follows that $h(a) \in T'$, thus $a \in T$ and $h(a) \in T''$. That is, $a' \in T''$. We have proved that $T' \subseteq T''$. Similarly, one shows $T'' \subseteq T'$. Hence, $T' = T''$ and h is stationary. □

Proposition 3.19. *If $h : \mathcal{L} \to \mathcal{L}'$ is a normal logic map, then*

$$A \Vdash_{\mathcal{L}} a \Longleftrightarrow h(A) \Vdash_{\mathcal{L}'} h(a)\,,$$

for all $A \cup \{a\} \subseteq Expr_{\mathcal{L}}$.

Proof. By Theorem 3.8, h has a complement, say H. The left to right implication we have already poved in Proposition 3.2. In order to prove the other implication, we consider theories as abstract models. So suppose $h(A) \Vdash_{\mathcal{L}'} h(a)$ and let T be any $\mathcal{L}$-theory such that $T \vDash_{\mathcal{L}} A$. We must show that $T \vDash_{\mathcal{L}} a$. By Proposition 3.12, $H(T) \vDash_{\mathcal{L}'} h(A)$. By hypothesis, $H(T) \vDash_{\mathcal{L}'} h(a)$. Again by Proposition 3.12, $T \vDash_{\mathcal{L}} a$. Since T was an arbitrary theory, the assertion follows. □

Definition 3.20. Let $h : Expr_{\mathcal{L}} \to Expr_{\mathcal{L}'}$ be a logic map. We define a function $G : Th_{\mathcal{L}'} \to Th_{\mathcal{L}}$ by

$$G(T') = T :\Longleftrightarrow h^{-1}(T') = T\,.$$

The function G is called the inverse complement of the logic map h.

Since h is a logic map, G is well-defined. Note that G is uniquely determined. The inverse complement G exists for any logic map (in contrast to complements, which exist only for normal logic maps). It is a continuous map between the respective spaces:

Proposition 3.21. *The inverse complement G of a logic map $h : Expr_{\mathcal{L}} \to Expr_{\mathcal{L}'}$ is a continuous map form the space of $\mathcal{L}'$ to the space of $\mathcal{L}$.*

Proof. G is continuous if G^{-1} sends basic open sets from the space of $\mathcal{L}$ to open sets of the space of $\mathcal{L}'$. In order to see this we show that $G^{-1}(a_1^{*\mathcal{L}} \cap \cdots \cap a_n^{*\mathcal{L}}) = h(a_1)^{*\mathcal{L}'} \cap \cdots \cap h(a_n)^{*\mathcal{L}}$, for any sequence $a_1, \ldots, a_n \in Expr_{\mathcal{L}}$ (recall that the basic opens $\neq \{\varnothing\}$ are exactly of the form $a_1^{*\mathcal{L}} \cap \cdots a_n^{*\mathcal{L}}$). If $\varnothing \in Th_{\mathcal{L}}$, then $\{\varnothing\}$ is a basic open and it is clear by the definition of G that $G^{-1}(\{\varnothing\}) = \{\varnothing\}$, which is again a basic open of the space of $\mathcal{L}'$. Since $G^{-1}(a^{*\mathcal{L}} \cap b^{*\mathcal{L}}) = G^{-1}(a^{*\mathcal{L}}) \cap G^{-1}(b^{*\mathcal{L}})$ for all $a, b \in Expr_{\mathcal{L}}$, it is sufficient to show that G satisfies the following, for each $a \in Expr_{\mathcal{L}}$: $G^{-1}(a^{*\mathcal{L}}) = h(a)^{*\mathcal{L}'}$:
We have $T' \in G^{-1}(a^{*\mathcal{L}})$ iff $G(T') = h^{-1}(T') \in a^{*\mathcal{L}}$ iff $a \in h^{-1}(T')$ iff $h(a) \in T'$ iff $T' \in h(a)^{*\mathcal{L}'}$. □

The next observation follows readily from the definitions:

Lemma 3.22. *Let $h : Expr_{\mathcal{L}} \to Expr_{\mathcal{L}'}$ be a logic map and let G be the inverse complement of h. h is normal if and only if G is surjective. h is stationary if and only if G is injective. Thus, G is a bijection iff h is a normal and stationary logic map. If G is a bijection, then $G = H^{-1}$, where H is the (unique) complement of h.*

The last assertion of the previous lemma says that the inverse complement G of a logic map h is in fact the inverse function of the (unique) complement H of h iff h is normal and stationary.

The following result connects our approach to the theory of Instituitons (see [5]). It follows easily from the definitions:

Corollary 3.23. *Let $h : Expr_{\mathcal{L}} \to Expr_{\mathcal{L}'}$ be a logic map and let G be the inverse complement of h. Then for all $T' \in Th_{\mathcal{L}'}$ and for all $a \in Expr_{\mathcal{L}}$ it holds the following:*

$$h(a) \in T' \Longleftrightarrow a \in G(T')\,.$$

If we write this condition interpreting theories as abstract models, we get:

$$T' \vDash_{\mathcal{L}'} h(a) \Longleftrightarrow G(T') \vDash_{\mathcal{L}} a\,.$$

The second condition in the above Corollary 3.23 has the same form as the Satisfaction Axiom of Institutions (see for instance [5]). It may be interesting to study in future works in detail the relationships between our approach to abstract logics and the concept of Institution.

4. Logic homomorphisms

In this chapter we study particular logic maps, called logic homomorphisms, that preserve more topological structure than general logic maps: we already know that the inverse complement of a logic map is continuous. In the following we will see that the inverse complement of a logic homomorphism is also an open map, besides beeing continuous. Furthermore, if a logic homomorphism h is normal and stationary, then its complement H and its inverse complement $G = H^{-1}$ turn out to be homeomorphisms. In this sense, a normal and stationary logic homomorphism establishes equivalence of two logics. This could be a reason to call such maps logic isomorphisms. Though, one requires from isomorphisms certain nice properties: they should be preserved under composition and inverse. However, a normal and stationary logic homomorphism is in general not surjective. So what is here an inverse or a composition? We will see that our notion of L-surjectivity (which is not too strong) helps to solve the problem. Then an L-surjective, normal (stationary) logic homomorphism defines a logic isomorphism, i.e., our concept of equivalence of logics.

Definition 4.1. Let $\mathcal{L}$ and $\mathcal{L}'$ be abstract logics. A function $h : Expr_{\mathcal{L}} \to Expr_{\mathcal{L}'}$ is called a logic homomorphism from $\mathcal{L}$ to $\mathcal{L}'$, if for every $\tau' \in B_{\mathcal{L}'}$ there exists some set $A_{\tau'} \subseteq B_{\mathcal{L}}$ such that the following holds:

$$h^{-1}(\tau') = \cup\big\{\tau \mid \tau \in A_{\tau'}\big\}\,.$$

In other words, h is a logic homomorphism, if the inverse of h sends open sets of the space of $\mathcal{L}'$ to open sets of the space of $\mathcal{L}$.[10]

[10]Notice that it is enough to require this property for basic opens of the space of $\mathcal{L}'$. Therefore, in the defining condition of a logic homomorphism, we only consider basic opens $\tau' \in B_{\mathcal{L}'}$.

Remark 4.2. Suppose the following: the logics $\mathcal{L}, \mathcal{L}'$ have conjunction and if $\varnothing \in Th_{\mathcal{L}} \cap Th_{\mathcal{L}'}$, then $\mathcal{L}, \mathcal{L}'$ have some inconsistent formula, respectively. Then, by Proposition 2.9, $B_{\mathcal{L}} = S_{\mathcal{L}}$ and $B_{\mathcal{L}'} = S_{\mathcal{L}'}$. Thus, if for each $a' \in Expr_{\mathcal{L}'}$ there is some $A_{a'} \subseteq Expr_{\mathcal{L}}$ such that

$$h^{-1}(a'^{*_{\mathcal{L}'}}) = \cup\big\{a^{*_{\mathcal{L}}} \mid a' \in A_{a'}\big\}\,,$$

then h is a logic homomorphism from $\mathcal{L}$ to $\mathcal{L}'$.

Proposition 4.3. *Logic homomorphisms are logic maps.*

Proof. Let $h : \mathcal{L} \to \mathcal{L}'$ be a logic homomorphism. We must show that $\{h^{-1}(T') \mid T' \in Th_{\mathcal{L}'}\} \subseteq Th_{\mathcal{L}}$. Let $T' \in Th_{\mathcal{L}'}$. First, we suppose that $T' \neq \varnothing$. Then for any $a' \in T'$, $T' \in a'^{*_{\mathcal{L}'}}$. Hence, $h^{-1}(T') \in h^{-1}(a'^{*_{\mathcal{L}'}}) = \cup\{\tau \mid \tau \in A_{a'}\}$, for some $A_{a'} \subseteq B_{\mathcal{L}}$. Then there is some $\tau \in A_{a'}$, such that $h^{-1}(T') \in \tau$. Thus, $h^{-1}(T')$ must be a theory in $Th_{\mathcal{L}}$.

Now let us consider the case $T' = \varnothing$. Clearly, $h^{-1}(T') = h^{-1}(\varnothing) = \varnothing$. Thus, we must show that the empty set is a theory of logic $\mathcal{L}$, $\varnothing \in Th_{\mathcal{L}}$. We argue in the following way: By the second defining axiom of abstract logics, $\varnothing = \cap\mathcal{T}'$, for some $\mathcal{T}' \subseteq Th_{\mathcal{L}'}$ such that $\varnothing \notin \mathcal{T}'$. Then $\varnothing = h^{-1}(\varnothing) = h^{-1}(\cap\mathcal{T}') = \cap h^{-1}(\mathcal{T}') = \cap\mathcal{T} \in Th_{\mathcal{L}}$, since $\mathcal{T} = \{h^{-1}(T'') \mid T'' \in \mathcal{T}'\}$ is, by the preceding case, a set of $\mathcal{L}$-theories. □

Definition 4.4. A logic homomorphism is called normal (stationary), if it is normal (stationary) as a logic map, respectively.

Corollary 4.5. *Let $\mathcal{L}, \mathcal{L}'$ be logics and $h : Expr_{\mathcal{L}} \to Expr_{\mathcal{L}'}$ an L-surjective function. If for each $a' \in Expr_{\mathcal{L}'}$ there is some $A_{a'} \subseteq B_{\mathcal{L}}$ such that*

$$h^{-1}(a'^{*_{\mathcal{L}'}}) = \cup\big\{\tau \mid \tau \in A_{a'}\big\}$$

holds, then h is a logic homomorphism from $\mathcal{L}$ to $\mathcal{L}'$.

Proof. If h satisfies the above equation, then one shows in a similar way as in the proof of Proposition 4.3 that h is a logic map. By Lemma 3.18, h is stationary. Now it is easy to see that $h^{-1}(a'^{*_{\mathcal{L}'}} \cap b'^{*_{\mathcal{L}'}}) = h^{-1}(a'^{*_{\mathcal{L}'}}) \cap h^{-1}(b'^{*_{\mathcal{L}'}})$, for any $a', b' \in Expr_{\mathcal{L}'}$. From this it follows that h is a logic homomorphism. □

Logic homomorphisms have the nice property that under certain conditions they lead directly to continuous and open maps between the respective topological spaces:

Proposition 4.6. (i) *Suppose that $\mathcal{L}, \mathcal{L}'$ are abstract logics and $\mathcal{L}'$ has conjunction. Let $h : \mathcal{L} \to \mathcal{L}'$ be a logic homomorphism. Then the inverse complement G of h is an open and continuous map from the space of $\mathcal{L}'$ to the space of $\mathcal{L}$.*

(ii) *Now suppose that $\mathcal{L}, \mathcal{L}'$ are arbitrary abstract logics and $h : \mathcal{L} \to \mathcal{L}'$ is a normal and stationary logic homomorphism. Then the* (*unique*) *complement H of h and the inverse complement $G = H^{-1}$ are homeomorphisms between the respective spaces.*

Proof. (i) We already know that G is continuous (Proposition 3.21). Since $\mathcal{L}'$ has conjunction, the basic opens are given by $a'^{*_{\mathcal{L}'}}$, for $a' \in Expr_{\mathcal{L}'}$ (and $\{\varnothing\}$, if the empty set is a theory). It is easy to verify that $G(a'^{*_{\mathcal{L}'}}) = h^{-1}(a'^{*_{\mathcal{L}'}})$ holds, for all $a' \in Expr_{\mathcal{L}'}$. Since h is a logic homomorphism, G is an open map.

(ii) Now let $\mathcal{L}, \mathcal{L}'$ be arbitrary logics and suppose that the logic homomorphism h is normal and stationary. Then Lemma 3.22 says that the inverse complement $G = H^{-1}$ is bijective, where H is the unique complement of h. Furthermore, by Proposition 3.21, we know that G is continuous. From Proposition 3.14 (iii) it follows that H is continuous, since h is a logic homomorphism and $H^{-1}(a'^{*_{\mathcal{L}'}} \cap b'^{*_{\mathcal{L}'}}) = H^{-1}(a'^{*_{\mathcal{L}'}}) \cap H^{-1}(b'^{*_{\mathcal{L}'}})$, for any $a', b' \in Expr_{\mathcal{L}'}$. Hence, $G = H^{-1}$ is also open, thus, an homeomorphism. □

Before we give an example of a logic homomorphism, let us show that this concept is strictly stronger than logic maps: logic homomorphisms preserve topological structure, this is in general not the case for logic maps (or translations). It follows an example of a logic map which is not a logic homomorphism:

Example 8. Let us consider the first order logics $\mathcal{L}$ and $\mathcal{L}'$ (over an given finite signature Σ) and the logic map $i : \mathcal{L} \mapsto \mathcal{L}'$ of Example 6. In $\mathcal{L}'$ the class of all infinite Σ-structures is axiomatizable by a single formula φ (take φ as the infinite conjunction of the formulas φ_n, $n \leq \omega$, expressing that there are at least n elements). We prove that there is no such formula in $\mathcal{L}$. Towards a contradiction, suppose that ψ is an $\mathcal{L}$-expression that holds in all infinite $\mathcal{L}$-structures. Then

$$\{\varphi_n \mid n \leq \omega\} \Vdash_{\mathcal{L}} \psi \,.$$

Since $\mathcal{L}$ is compact, there is some $n' < \omega$ and

$$\{\varphi_n \mid n \leq n'\} \Vdash_{\mathcal{L}} \psi \,.$$

This means that ψ also holds in all structures with at least n' elements, in particular, ψ holds in (almost all) finite structures. Thus, ψ can not axiomatize the class of all infinite structures. In fact, similarly one shows that no class of infinite structures is axiomatizable by a single formula in $\mathcal{L}$.

Now we conclude that $i^{-1}(\varphi^{MTh_{\mathcal{L}'}}) = \{i^{-1}(T') \mid T' \in \varphi^{MTh_{\mathcal{L}'}}\} = \{T \in MTh_{\mathcal{L}} \mid T$ is the theory of an infinite model $\}$ can not be an open set of the topological subspace induced by $MTh_{\mathcal{L}} \subseteq Th_{\mathcal{L}}$: it can not be a basic open $\psi^{MTh_{\mathcal{L}}}$, since we have already shown that such a ψ does not exist. Neither it can be an open of the form $\cup\{\psi^{MTh_{\mathcal{L}}} \mid \psi \in X\}$, for some $X \subseteq Expr_{\mathcal{L}}$. Otherwise, each ψ would axiomatize a class of infinite structures, which is also impossible by a similar argumentation as above. We show that $i^{-1}(\varphi^{*_{\mathcal{L}'}})$ is not an open in the space of $\mathcal{L}$. Towards a contradiction suppose that $i^{-1}(\varphi^{*_{\mathcal{L}'}}) = \mathcal{O}$ is open. Then $\mathcal{O} \cap MTh_{\mathcal{L}} = i^{-1}(\varphi^{MTh_{\mathcal{L}'}})$ is an open in the subspace induced by $MTh_{\mathcal{L}}$, a contradiction. Hence, $i : \mathcal{L} \to \mathcal{L}'$ is a logic map but not a logic homomorphism.

The absence of a logic homomorphism in the above example expresses the fact that $\mathcal{L}'$ has a strictly stronger expressive power that $\mathcal{L}$.

As we shall see later, the following definition yields examples of normal, stationary logic homomorphisms:

Definition 4.7. A logic homomorphism $h : \mathcal{L} \to \mathcal{L}'$ is called strong, if for all $a' \in Expr_{\mathcal{L}'}$ the following holds:

$$h^{-1}(a'^{*_{\mathcal{L}'}}) = \cup\big\{A^{*_{\mathcal{L}}} \mid A \subseteq Expr_{\mathcal{L}} \text{ is finite and } h(A)^{*_{\mathcal{L}'}} \subseteq a'^{*_{\mathcal{L}'}}\big\}\,.$$

Remark 4.8. Notice that the defining condition of a strong logic homomorphism is equivalent to the condition

$$h^{-1}(a'^{*_{\mathcal{L}'}}) = \cup\big\{A^{*_{\mathcal{L}}} \mid A \text{ is finite and } h(A) \Vdash_{\mathcal{L}'} a'\big\}\,.$$

If $A = \{a_1, \ldots, a_n\}$, then $A^{*_{\mathcal{L}}} = a_1^{*_{\mathcal{L}}} \cap \cdots \cap a_n^{*_{\mathcal{L}}}$. Furthermore, we will see that strong logic homomorphisms are stationary. From this it follows that $h^{-1}(a'^{*_{\mathcal{L}'}} \cap b'^{*_{\mathcal{L}'}}) = h^{-1}(a'^{*_{\mathcal{L}'}}) \cap h^{-1}(b'^{*_{\mathcal{L}'}})$, for any $a', b' \in Expr_{\mathcal{L}'}$. Thus, the inverse of a strong logic homomorphism h sends arbitrary basic open sets to open sets.

Proposition 4.9. *Strong logic homomorphisms are normal and stationary.*

Proof. Let $h : \mathcal{L} \to \mathcal{L}'$ be a strong logic homomorphism. For $a \in Expr_{\mathcal{L}}$ we have $a^{*_{\mathcal{L}}} \subseteq \cup\{A^{*_{\mathcal{L}}} \mid A \text{ is finite and } h(A)^{*_{\mathcal{L}'}} \subseteq h(a)^{*_{\mathcal{L}'}}\} = h^{-1}(h(a)^{*_{\mathcal{L}'}})$. Now, from Proposition 3.2 (iv) and Lemma 3.5 it follows that h is normal. In order to prove that h is stationary suppose that $T'_1, T'_2 \in Th_{\mathcal{L}'}$ and $h^{-1}(T'_1) = h^{-1}(T'_2)$. Let $a' \in T'_1$, that is, $T'_1 \in a'^{*_{\mathcal{L}'}}$. Then $T := h^{-1}(T'_1) \in h^{-1}(a'^{*_{\mathcal{L}'}}) = \cup\{A^{*_{\mathcal{L}}} \mid A \text{ is finite and } h(A) \Vdash_{\mathcal{L}'} a'\}$. Thus, $T \in A^{*_{\mathcal{L}}}$, for some finite A with $h(A) \Vdash_{\mathcal{L}'} a'$. Since $A \subseteq T$, $h(A) \subseteq h(T) \subseteq T'_2$. T'_2 is deductively closed and it follows that $a' \in T'_2$. Hence, $T'_1 \subseteq T'_2$. The inclusion $T'_2 \subseteq T'_1$ follows in the same way. □

The next result follows immediately from Propositions 4.6 and 4.9:

Corollary 4.10. *If there is a strong logic homomorphism from $\mathcal{L}$ to $\mathcal{L}'$, then the underlying topological spaces are homeomorphic.*

An important concept of relationship between logics is the concept of extension. Extensions of logics are a rather common phenomenon in practice. Simple examples are given in first order logic by adding new non-logical symbols to a given language. A suitable interpretation of the so extended language yields an "extension". On the other hand, one also gets extensions by extending the logical part of the language. Such cases one can find in some of our examples given above. In order to provide a precise formulation of this intuitive concept we adapt a definition of [8] (page 37) to our approach.

Definition 4.11. Suppose that $\mathcal{L}, \mathcal{L}'$ are abstract logics with $Expr_{\mathcal{L}} \subseteq Expr_{\mathcal{L}'}$. $\mathcal{L}'$ is called an extension of $\mathcal{L}$, if $Th_{\mathcal{L}} = \{T' \cap Expr_{\mathcal{L}} \mid T' \in Th_{\mathcal{L}'}\}$. We write $\mathcal{L} \leq \mathcal{L}'$. $\mathcal{L}'$ is a definable extension of $\mathcal{L}$, if $\mathcal{L} \leq \mathcal{L}'$ and for each $a' \in Expr_{\mathcal{L}'}$ there is some $a \in Expr_{\mathcal{L}}$ such that $a' =_{\mathcal{L}'} a$. We write $\mathcal{L} \leq_{def} \mathcal{L}'$.

We are able to prove the following connection between extensions and our logic maps:

Lemma 4.12. *Let $\mathcal{L}, \mathcal{L}'$ be abstract logics with $Expr_{\mathcal{L}} \subseteq Expr_{\mathcal{L}'}$.*

(i) *The identity map $i : Expr_{\mathcal{L}} \to Expr_{\mathcal{L}'}$, $a \mapsto a$, is a normal logic map if and only if $\mathcal{L} \leq \mathcal{L}'$.*

(ii) *The identity map $i : Expr_{\mathcal{L}} \to Expr_{\mathcal{L}'}$ is a normal, L-surjective strong logic homomorphism if and only if $\mathcal{L} \leq_{def} \mathcal{L}'$.*

Proof. (i) Note that for $T' \in Th_{\mathcal{L}'}$, $i^{-1}(T') = T' \cap Expr_{\mathcal{L}}$. Now (i) follows easily from the definitions.

(ii) Suppose that the identity map is a normal, L-surjective strong logic homomorphism. By (i), $\mathcal{L} \leq \mathcal{L}'$. By L-surjectivity of i, $\mathcal{L} \leq_{def} \mathcal{L}'$.

Now let $\mathcal{L} \leq_{def} \mathcal{L}'$. By (i), i is a normal logic map. i is also L-surjective, since the extension is definable. We show that i is a logic homomorphism. Let $a' \in Expr_{\mathcal{L}'}$. There is some $a \in Expr_{\mathcal{L}}$ such that $a =_{\mathcal{L}'} a'$. Then $i^{-1}(a'^{*_{\mathcal{L}'}}) = i^{-1}(a^{*_{\mathcal{L}'}}) = a^{*_{\mathcal{L}}}$. Hence, i is a logic homomorphism. In order to show that i is a strong logic homomorphism it suffices to prove that $a^{*_{\mathcal{L}}} = \cup\{A^{*_{\mathcal{L}}} \mid A \subseteq Expr_{\mathcal{L}}$ is finite and $i(A)^{*_{\mathcal{L}'}} \subseteq a'^{*_{\mathcal{L}'}}\}$. The inclusion " $\subseteq$" is evident, since $i(a)^{*_{\mathcal{L}'}} = a^{*_{\mathcal{L}'}} = a'^{*_{\mathcal{L}'}}$. Assume that $A \subseteq Expr_{\mathcal{L}}$ is finite and $i(A)^{*_{\mathcal{L}'}} \subseteq a'^{*_{\mathcal{L}'}} = a^{*_{\mathcal{L}'}}$. Then $A^{*_{\mathcal{L}}} = A^{*_{\mathcal{L}'}} \cap Th_{\mathcal{L}} \subseteq a^{*_{\mathcal{L}'}} \cap Th_{\mathcal{L}} = a^{*_{\mathcal{L}}}$, and the inclusion " $\supseteq$" follows. □

Theorem 4.13. *Suppose that $\mathcal{L}, \mathcal{L}'$ are abstract logics. Let $h : Expr_{\mathcal{L}} \to Expr_{\mathcal{L}'}$ be a function. The following statements are equivalent:*

(i) *h is an L-surjective normal logic map.*

(ii) *h is an L-surjective normal logic homomorphism.*

(iii) *h is an L-surjective strong logic homomorphism.*

(iv) *h is an L-surjective normal and stationary logic map.*

Proof. (i)→(ii) Suppose that h is a normal and L-surjective logic map. Let $a' \in Expr_{\mathcal{L}'}$. Since h is L-surjective, there is some $a \in Expr_{\mathcal{L}}$ such that $h(a) =_{\mathcal{L}'} a'$. (Note that $a'^{*_{\mathcal{L}'}} = h(a)^{*_{\mathcal{L}'}}$.) Then from Proposition 3.14 (i) it follows that $h^{-1}(a'^{*_{\mathcal{L}'}}) = a^{*_{\mathcal{L}}}$. In particular, h satisfies the equation of Corollary 4.5 (put $A_{a'} := \{a^{*_{\mathcal{L}}}\}$). Then this Corollary says that h is a logic homomorphism.

(ii)→(iii): Now suppose that h is a normal and L-surjective logic homomorphism. Let $a' \in Expr_{\mathcal{L}'}$. In the same way as above we get $h^{-1}(a'^{*_{\mathcal{L}'}}) = a^{*_{\mathcal{L}}}$, where $h(a) =_{\mathcal{L}'} a'$. We show that h is also strong: It suffices to prove that $a^{*_{\mathcal{L}}} = \cup\{A^{*_{\mathcal{L}}} \mid A$ is finite and $h(A)^{*_{\mathcal{L}'}} \subseteq h(a)^{*_{\mathcal{L}'}}\}$. Then, by definition, h must be a strong logic homomorphism.

Cearly, $a^{*_{\mathcal{L}}} \in \{A^{*_{\mathcal{L}}} \mid A$ is finite and $h(A)^{*_{\mathcal{L}'}} \subseteq h(a)^{*_{\mathcal{L}'}}\}$ (consider $A = \{a\}$). Now suppose $T \in \cup\{A^{*_{\mathcal{L}}} \mid A$ is finite and $h(A)^{*_{\mathcal{L}'}} \subseteq h(a)^{*_{\mathcal{L}'}}\}$. Then $T \in A^{*_{\mathcal{L}}}$, for some finite A such that $h(A)^{*_{\mathcal{L}'}} \subseteq h(a)^{*_{\mathcal{L}'}}$. Since h is normal, by Lemma 3.5 we get $A^{*_{\mathcal{L}}} = h^{-1}(h(A)^{*_{\mathcal{L}'}}) \subseteq h^{-1}(h(a)^{*_{\mathcal{L}'}}) = a^{*_{\mathcal{L}}}$. Hence, $A^{*_{\mathcal{L}}} \subseteq a^{*_{\mathcal{L}}}$, and the assertion follows.

The implication (iii)→(iv) follows from the previous results. Finally, (iv)→(i) is trivial. □

Definition 4.14. Let $\mathcal{L}, \mathcal{L}'$ be abstract logics. A function $h : Expr_{\mathcal{L}} \to Expr_{\mathcal{L}'}$ is called a logic isomorphism, if h satisfies one of the equivalent conditions of Theorem 4.13. If there is some logic isomorphism from $\mathcal{L}$ to $\mathcal{L}'$, then we say that the logics $\mathcal{L}$ and $\mathcal{L}'$ are isomorphic.

As a consequence of Lemma 4.12 we obtain that all definable extensions of some abstract logic $\mathcal{L}$ are isomorphic to $\mathcal{L}$. In other words, an abstract logic has up to isomorphism only one definable extension.

We give an example of a strong logic homomorphism which is not an isomorphism. It is an adaption of an example from [4].

Example 9. We return to the logics $\mathcal{L}, \mathcal{L}'$ of our Examples 8 and 6. We have seen that the identity function $i : Expr_{\mathcal{L}} \to Expr_{\mathcal{L}'}$ is a logic map but not a logic homomorphism. In these preceding examples $\mathcal{L}$ and $\mathcal{L}'$ were logics generated by the class of all Σ-structure (for some given finite first order signature Σ.) Here we assume the respective sublogics (see Definition 2.1) which are generated by the class of all finite Σ-structures. That is, we assume that $\mathcal{L}$ is the abstract logic with classical first order expressions over Σ generated by all finite Σ-structures, and $\mathcal{L}'$ is the abstract logic with $\mathcal{L}_{\omega_1\omega}$-expressions over Σ generated by all finite Σ-structures.

It is well-known that every finite model $\mathcal{A}$ is characterizable – up to ismorphism - by a single first order formula, that is, by a formula $\varphi \in Expr_{\mathcal{L}}$.[11] This formula φ isolates the theory T of the finite model $\mathcal{A}$. That is, $\Vdash_{\mathcal{L}} \varphi \to \psi$ holds iff $\psi \in T$. In other words, for each complete (=maximal) theory $T \in MTh_{\mathcal{L}}$ there is a formula $\varphi \in T$ such that $\varphi^{MTh_{\mathcal{L}}} = \{T\}$ holds: T is an isolated point in the topological subspace induced by $MTh_{\mathcal{L}} \subseteq Th_{\mathcal{L}}$. It follows that $\varphi^{*_{\mathcal{L}}} = \{T\}$, that is, φ isolates T also in the space of $\mathcal{L}$. For every complete theory T of logic $\mathcal{L}$ and every complete theory T' of logic $\mathcal{L}'$ such that $T = T' \cap Expr_{\mathcal{L}}$ it is easy to see that $\varphi \in Expr_{\mathcal{L}}$ isolates T in the space of $\mathcal{L}$ iff φ isolates T' in the space of $\mathcal{L}'$. Then for every $\varphi' \in Expr_{\mathcal{L}'}$ it holds the following:

$$\begin{aligned} i^{-1}(\varphi'^{MTh_{\mathcal{L}'}}) &= \big\{ i^{-1}(T') \mid \varphi' \in T' \in MTh_{\mathcal{L}'} \big\} \\ &= \big\{ T' \cap Expr_{\mathcal{L}} \mid \varphi' \in T' \in MTh_{\mathcal{L}'} \big\} \\ &= \cup \big\{ \psi^{MTh_{\mathcal{L}}} \mid \psi \in Expr_{\mathcal{L}} \text{ isolates some } T' \in \varphi'^{*_{\mathcal{L}'}} \big\} \\ &\overset{(*)}{=} \cup \big\{ \psi^{MTh_{\mathcal{L}}} \mid i(\psi) \Vdash_{\mathcal{L}'} \varphi' \big\} \\ &= \cup \big\{ \psi^{MTh_{\mathcal{L}}} \mid i(\psi)^{MTh_{\mathcal{L}'}} \subseteq \varphi'^{MTh_{\mathcal{L}'}} \big\}. \end{aligned}$$

We show the equation (*): Let $T \in \cup\{\psi^{MTh_{\mathcal{L}}} \mid \psi \in Expr_{\mathcal{L}}$ isolates some $T' \in \varphi'^{*_{\mathcal{L}'}}\}$. Then $T \in \psi^{MTh_{\mathcal{L}}}$ for some ψ that isolates some $T' \in \varphi'^{*_{\mathcal{L}'}}$. Thus, ψ also isolates $T \subseteq T'$. It is also clear that $i(\psi) = \psi \Vdash_{\mathcal{L}'} \varphi'$, thus the left-to-right-implication holds. Now suppose $T \in \cup\{\psi^{MTh_{\mathcal{L}}} \mid i(\psi) \Vdash_{\mathcal{L}'} \varphi'\}$. That is, there is some $\psi \in Expr_{\mathcal{L}}$ such that $\psi \in T$ and $\psi \Vdash_{\mathcal{L}'} \varphi'$. Consider $T' \in MTh_{\mathcal{L}'}$ with

[11] The isomorphism type of a finite model $\mathcal{A}$ is uniquely given by the diagram of $\mathcal{A}$, which is finitely axiomatizable, i.e., by a single formula.

$T \subseteq T'$ (T' is the complete $\mathcal{L}'$-theory of $\mathcal{A}$, where $\mathcal{A}$ is a model of T in $\mathcal{L}$). $\psi \Vdash_{\mathcal{L}'} \varphi'$ implies that $\varphi' \in T'$. There is some $\chi \in Expr_{\mathcal{L}} \cap T'$ which isolates T' in $\mathcal{L}'$. Then χ also isolates T in $\mathcal{L}$. Since $T' \in \varphi'^{*_{\mathcal{L}'}}$, the implication from right to left follows.

Now it follows easily $i^{-1}(\varphi'^{*_{\mathcal{L}'}}) = \cup\{\psi^{*_{\mathcal{L}}} \mid i(\psi)^{*_{\mathcal{L}'}} \subseteq \varphi'^{*_{\mathcal{L}'}}\}$. Thus, i is a strong logic homomorphism. However, i is not L-surjective: For example, if Σ is the language of graphs, then connectivity of (finite) graphs is axiomatizable by a single $\mathcal{L}'$-sentence.[12] This is not possible by an $\mathcal{L}$-sentence. Therefore, the two logics can not be isomorphic.

Let us summarize:

- The identity map i is a strong logic homomorphism from $\mathcal{L}$ to $\mathcal{L}'$. By Proposition 4.9, i is also normal and stationary.
- i is not a logic isomorphism, since i is not L-surjective.
- By Corollary 4.10, the underlying spaces are homeomorphic. More precisely, let I be the (unique) complement of i. I is bijective. By Proposition 4.6, I is an homeomorphism from the space of $Th_{\mathcal{L}}$ to the space of $Th_{\mathcal{L}'}$.

This example shows in particular that the underlying topological spaces of two logics $\mathcal{L}$ and $\mathcal{L}'$ may be homeomorphic although $\mathcal{L}$ and $\mathcal{L}'$ are not isomorphic. Thus, the existence of a logic isomorhism is a sufficient but not a necessary condition for the existence of an homeomorphism between the respective spaces.

Usually, the notion "isomorphic" gives rise to an equivalence relation. This means in our case that if $h : \mathcal{L} \to \mathcal{L}'$ is a logic isomorphism, we expect that there exists also a logic isomorphism from $\mathcal{L}'$ to $\mathcal{L}$. Furthermore, if $h_1 : \mathcal{L} \to \mathcal{L}'$ and $h_2 : \mathcal{L}' \to \mathcal{L}''$ are logic isomorphisms, then there should exist a logic isomorphism $h_3 : \mathcal{L} \to \mathcal{L}''$. Indeed, the following results guarantee that this is true.

Theorem 4.15. *Let $h : \mathcal{L} \to \mathcal{L}'$ be a logic isomorphism. Then there exists a logic isomorphism $h' : \mathcal{L}' \to \mathcal{L}$ such that for all $a \in Expr_{\mathcal{L}}$ and all $a' \in Expr_{\mathcal{L}'}$ the following holds:*

(i) $h'(h(a)) =_{\mathcal{L}} a$, *and*
(ii) $h(h'(a')) =_{\mathcal{L}'} a'$.

Proof. Since h is L-surjective, we may choose for each $a' \in Expr_{\mathcal{L}'}$ some $a \in Expr_{\mathcal{L}}$ such that $h(a) =_{\mathcal{L}'} a'$. Now we define $h' : Expr_{\mathcal{L}'} \to Expr_{\mathcal{L}}$ by $a' \mapsto a$.

In order to prove that h' is a logic isomorphism, we show that h' is a normal and L-surjective logic map.

Suppose $a' \neq_{\mathcal{L}'} b'$, $h^{-1}(a'^{*_{\mathcal{L}'}}) = a^{*_{\mathcal{L}}}$ and $h^{-1}(b'^{*_{\mathcal{L}'}}) = b^{*_{\mathcal{L}}}$. That is, $h'(a') =_{\mathcal{L}} a$ and $h'(b') =_{\mathcal{L}} b$. Since $a' \neq_{\mathcal{L}'} b'$, there is some theory $T' \in Th_{\mathcal{L}'}$ such that $a' \in T'$ and $b' \notin T'$. Then $h^{-1}(T') = T \in Th_{\mathcal{L}}$ and $a \in T$ and $b \notin T$. Therefore, $a \neq_{\mathcal{L}} b$. Thus, h' is an L-injective function.

We show that (i) and (ii) hold: Let $a \in Expr_{\mathcal{L}}$ and suppose $h'(h(a)) = b$ for some $b \in Expr_{\mathcal{L}}$. By definition of h', $h(b) =_{\mathcal{L}'} h(a)$. Since h is L-injective,

[12] For $n < \omega$ let φ_n be a formula expressing that x and y are connected by a chain of n elements. Now consider the sentence $\forall x \forall y \bigvee\{\varphi_n \mid n < \omega\}$.

we get $a =_{\mathcal{L}} b$, hence $h'(h(a)) =_{\mathcal{L}} a$ and (i) holds. Let $a' \in Expr_{\mathcal{L}'}$ and suppose $h(h'(a')) = b' \in Expr_{\mathcal{L}'}$. Let $h'(b') = c$. By definition, $h(c) =_{\mathcal{L}'} b'$. Thus, $h(h'(a')) =_{\mathcal{L}'} h(c)$ and L-injectivity of h yields $h'(a') =_{\mathcal{L}} c = h'(b')$. Then $a' =_{\mathcal{L}'} b'$, since h' is L-injective. Hence, (ii) holds.

From (i) it follows that h' is L-surjective.

Next, we prove that h' is a logic map. That is, we show that for any $T \in Th_{\mathcal{L}}$, $h'^{-1}(T) \in Th_{\mathcal{L}'}$. For any $a \in Expr_{\mathcal{L}}$ we have $h'^{-1}(a) = \{a' \mid h'(a') = a\} = \{a' \mid a' =_{\mathcal{L}'} h(a)\}$. Hence, $h'^{-1}(T) = \{a' \mid h'(a') \in T\} = \{a' \mid a' =_{\mathcal{L}'} b', b' \in h(T)\}$. Since h is normal, $h(T) \subseteq T'$ for some $T' \in Th_{\mathcal{L}'}$. Thus, $\{a' \mid a' =_{\mathcal{L}'} b', b' \in h(T)\}$ is consistent. Suppose $\{a' \mid a' =_{\mathcal{L}'} b', b' \in h(T)\} \Vdash_{\mathcal{L}'} c'$. It is easy to see that this implies $\{a' \mid a' \in h(T)\} \Vdash_{\mathcal{L}'} c'$. There is some $d' =_{\mathcal{L}'} c'$ such that $h(d) = d'$ for some $d \in Expr_{\mathcal{L}}$. Hence, $h(T) \Vdash_{\mathcal{L}'} d'$. By Proposition 3.19, $T \Vdash_{\mathcal{L}} d$. Hence, $d \in T$ and therefore $d' \in h(T)$ and $c' \in \{a' \mid a' =_{\mathcal{L}'} b', b' \in h(T)\}$. We have shown that $h'^{-1}(T) = \{a' \mid a' =_{\mathcal{L}'} b', b' \in h(T)\}$ is consistent and deductively closed, that is, it is a theory. Thus, h' is a logic map.

Finally, let us show that h' is normal. We have the following for each theory $T' \in Th_{\mathcal{L}'}$:

$$\begin{aligned} h'(T') &= \{h'(a') \mid a' \in T'\} \\ &\subseteq \{a \mid h(a) =_{\mathcal{L}'} a', a' \in T'\} \\ &= \{a \mid h(a) \in T'\} \\ &= h^{-1}(T') \in Th_{\mathcal{L}}\,. \end{aligned}$$

It follows that for each $T' \in Th_{\mathcal{L}'}$ there is some $T \in Th_{\mathcal{L}}$ such that $h'(T') \subseteq T$. We may assume that T is the smallest theory such that $h'(T') \subseteq T$ (by taking intersections of theories, i.e., the deductive closure of $h'(T')$). We prove that $h'^{-1}(T) = T'$ holds. We know that $T' \subseteq h'^{-1}(T)$. Now let $b' \in h'^{-1}(T) \smallsetminus T'$. Thus, $h'(b') \in T$. Since T is the smallest theory containing $h'(T')$, it follows that every theory that contains $h'(T')$ contains also $h'(b')$. That is, $h'(T') \Vdash_{\mathcal{L}} h'(b')$. Since h is a logic map, Proposition 3.2 yields $h(h'(T')) \Vdash_{\mathcal{L}'} h(h'(b'))$. Hence, $T' \Vdash_{\mathcal{L}'} b'$, since $h(h'(T')) \subseteq T'$, by (ii). Since T' is a theory, it is deductively closed. Thus, $b' \in T'$ and $h'^{-1}(T) = T'$.

We have proved that for each $T' \in Th_{\mathcal{L}'}$ there is some $T \in Th_{\mathcal{L}}$ such that $h'(T') \subseteq T$ and $h'^{-1}(T) = T'$. Summarizing, $h' : \mathcal{L}' \to \mathcal{L}$ is a normal L-surjective logic map, i.e., a logic isomorphism. □

Theorem 4.16. *If $h_1 : \mathcal{L} \to \mathcal{L}'$ and $h_2 : \mathcal{L}' \to \mathcal{L}''$ are logic isomorphisms, then there exists a logic isomorphism $h_3 : \mathcal{L} \to \mathcal{L}''$.*

Proof. Suppose that the hypothesis holds. We define $h_3 : Expr_{\mathcal{L}} \to Expr_{\mathcal{L}''}$ by $a \mapsto h_2(h_1(a))$.

Let $a'' \in Expr_{\mathcal{L}''}$. Since h_1 and h_2 are L-surjective, there are $a' \in Expr_{\mathcal{L}'}$ and $a \in Expr_{\mathcal{L}}$ such that $h_2(a') =_{\mathcal{L}''} a''$ and $h_1(a) =_{\mathcal{L}'} a'$. Note that h_1 and h_2

are normal. Then Proposition 3.5 yields

$$\begin{aligned} h_3^{-1}(a''^{*_{\mathcal{L}''}}) &= h_1^{-1}\Big(h_2^{-1}\big(h_2(a')^{*_{\mathcal{L}''}}\big)\Big) \\ &= h_1^{-1}(a'^{*_{\mathcal{L}'}}) \\ &= h_1^{-1}\big(h_1(a)^{*_{\mathcal{L}'}}\big) \\ &= a^{*_{\mathcal{L}}} . \end{aligned}$$

Thus, $h_3^{-1}(h_3(a)^{*_{\mathcal{L}''}}) = a^{*_{\mathcal{L}}}$. Applying again Proposition 3.5 we obtain that h_3 is a normal L-surjective logic homomorphism, i.e., a logic isomorphism. □

Definition 4.17. Let $\mathcal{L} = (Expr_{\mathcal{L}}, Th_{\mathcal{L}})$ be an abstract logic. Let $\equiv$ be an equivalence relation on $Expr_{\mathcal{L}}$ that refines $\mathcal{L}$-equivalence. We denote the equivalence classes of $Expr_{\mathcal{L}}$ modulo $\equiv$ by $[a]_{\equiv}$, for $a \in Expr_{\mathcal{L}}$. The set of all equivalence classes is denoted by $Expr_{\mathcal{L}_{\equiv}}$. For each $T \in Th_{\mathcal{L}}$ we write $T_{\equiv}$ for the set $\{[a]_{\equiv} \mid a \in T\}$. $Th_{\mathcal{L}_{\equiv}}$ denotes the set $\{T_{\equiv} \mid T \in Th_{\mathcal{L}}\}$. Then the factor logic of $\mathcal{L}$ modulo $\equiv$ is defined by $\mathcal{L}_{\equiv} = (Expr_{\mathcal{L}_{\equiv}}, Th_{\mathcal{L}_{\equiv}})$.

The notions of the above definition are well-defined:

(i) $a \in T$ and $a \equiv b$ implies $b \in T$, for all theories T and all expressions a, b.
(ii) $(T_1 \cap T_2)_{\equiv} = T_{1\equiv} \cap T_{2\equiv}$, for all theories $T_1, T_2 \in Th_{\mathcal{L}}$.

The concept of factor logic provides particular examples of logic isomorphisms:

Proposition 4.18. *Let $\mathcal{L}$ be an abstract logic and $\equiv$ an equivalence relation that refines $\mathcal{L}$-equivalence. Then $\mathcal{L}$ and its factor logic $\mathcal{L}_{\equiv}$ are isomorphic.*

Proof. We define $h : Expr_{\mathcal{L}} \to Expr_{\mathcal{L}_{\equiv}}$ by $a \mapsto [a]_{\equiv}$. It is clear that $h^{-1}(T_{\equiv}) = T$ and $h(T) = T_{\equiv}$, for all $T \in Th_{\mathcal{L}}$. Thus, h is a normal and L-surjective logic map. That is, it is a logic isomorphism. □

Let $h : \mathcal{L} \to \mathcal{L}'$ be an L-injective and L-surjective logic map. Let $\equiv$ be $\mathcal{L}$-equivalence and let $\equiv'$ be $\mathcal{L}'$-equivalence. Then it is easy to see that the function h' defined by $[a]_{\equiv} \mapsto [h(a)]_{\equiv'}$ is a bijective logic map from the factor logic $\mathcal{L}_{\equiv}$ to the factor logic $\mathcal{L}'_{\equiv'}$. Furthermore, h' normal if and only if h is normal.

Particular examples of equivalence relations that refine $\mathcal{L}$-equivalence are given by kernels of L-injective functions:

Let $\mathcal{L}, \mathcal{L}'$ be abstract logics and $h : Expr_{\mathcal{L}} \to Expr_{\mathcal{L}'}$ an L-injective function. Put $Ker_h := \{(a, b) \in Expr_{\mathcal{L}} \mid h(a) = h(b)\}$. It is clear that Ker_h is an equivalence relation. Suppose that $a \neq_{\mathcal{L}} b$. Then $h(a) \neq_{\mathcal{L}'} h(b)$, since h is L-injective. In particular, $h(a) \neq h(b)$ and $(a, b) \notin Ker_h$. Hence, Ker_h refines $\mathcal{L}$-equivalence.

In a final example we show that the sentential part of each classical first order logic (over some given signature Σ) is isomorphic to a sublogic of classical propositional (sentential) logic:

Example 10. Let Σ be a fixed first order signature and let $\mathcal{L} = (Expr_{\mathcal{L}}, Th_{\mathcal{L}})$ be the classical first order logic over Σ as introduced in Example 1. We consider the set

$Sen_{\mathcal{L}} \subseteq Expr_{\mathcal{L}}$ of sentences, that is, all first order formulas with no free variables. Then let $\mathcal{L}_S = (Sen_{\mathcal{L}}, Th_{\mathcal{L}_S})$, where $Th_{\mathcal{L}_S} = Th_{\mathcal{L}} \cap Sen_{\mathcal{L}}$. The logic $\mathcal{L}_S$ considers only the sentential part of classical first order logic over Σ. A atomic sentence is either a formula with no variables or it is a sentence of the form $(Qx.(\varphi))$, where Q is a quantifier and φ is a formula with at most x as free variable. Let $ASen_{\mathcal{L}}$ be the set of all atomic sentences. We assume a bijective function $h : ASen_{\mathcal{L}} \to P$ to some infinite set of propositional variables P. h extends to a function h' defined on all sentences:

$$\begin{aligned}
h'(\psi) &= h(\psi), \text{ if } \psi \text{ is atomic sentence} \\
h'(\psi_1 \vee \psi_2) &= h'(\psi_1) \vee h'(\psi_2) \\
h'(\psi_1 \wedge \psi_2) &= h'(\psi_1) \wedge h'(\psi_2) \\
h'(\psi_1 \to \psi_2) &= h'(\psi_1) \to h'(\psi_2) \\
h'(\sim \psi) &= \sim h'(\psi) .
\end{aligned}$$

(Note that in order to simplify matters we do not distinguish sintactically between the connectivities of the respective logics. We also write in the following h for h'.)

Let $\mathcal{L}_{\mathcal{P}} = (Expr_{\mathcal{L}_{\mathcal{P}}}, Th_{\mathcal{L}_{\mathcal{P}}})$ be the classical propositional logic over P. Then one easily verifies that $h : Sen_{\mathcal{L}} \to Expr_{\mathcal{L}_{\mathcal{P}}}$ is a bijective function. We show that h is a logic isomorphism to a sublogic of $\mathcal{L}_{\mathcal{P}}$.

For each maximal (=complete) theory $T \in MTh_{\mathcal{L}_S}$ we define a variable assignment $v_T : P \to \{0,1\}$ by $v_T(p) = 1$, if $h^{-1}(p) \in T$, and $v_T(p) = 0$ otherwise. This yields an embedding $F : MTh_{\mathcal{L}_S} \to 2^P$, given by $T \mapsto v_T$. Clearly, F can not be surjective.

Recall that the classical propositional logic $\mathcal{L}_{\mathcal{P}}$ is generated by the set of all variable assignments 2^P. Now we consider the (proper) propositional sublogic $\mathcal{L}' \subset \mathcal{L}_{\mathcal{P}}$ which is generated by the set $\{v_T \mid T \in MTh_{\mathcal{L}_S}\} \subset 2^P$. Then it is easy to see that h gives rise to a bijection $T \mapsto T'$ between $MTh_{\mathcal{L}_S}$ and $MTh_{\mathcal{L}'}$ given by $\varphi \in T \iff h(\varphi) \in T' \iff v_T(h(\varphi)) = 1$ (note that a variable assignment v_T extends canonically to a function defined on all propositional formulas). That is, $MTh_{\mathcal{L}'}$ is given by the set of all sets of the form $T' = \{a \in Expr_{\mathcal{L}_{\mathcal{P}}} \mid v_T(a) = 1\}$, for $T \in MTh_{\mathcal{L}_S}$. Therefore, h is also a bijection between $Th_{\mathcal{L}_S}$ and $Th_{\mathcal{L}'}$. (More exactly, this bijection is the unique complement H of h.) It is also clear that h is a L-surjective normal logic map. Hence, $h : \mathcal{L}_S \to \mathcal{L}'$ is a logic isomorphism.

Finally, we show that our concept of a logic isomorphism between abstract logics is - under certain restrictions - equivalent to the notion “equipollence of logical systems” as introduced in [2]. This is expressed in our Theorem 4.16.

In order to show the equivalence of the two proposals, we need a unique concept of “logic”. This is managed by restricting our notion of abstract logics to the syntactical presuppositions required in [2]. More precisely, we assume here the following:

(i) The expressions of an abstract logic are constructed over a given signature Σ as in Definition 2.1 of [2]. That is, the expressions are Σ-formulas in the sense of [2].
(ii) Every logic map $h : \mathcal{L} \to \mathcal{L}'$ is a uniform translation function which is induced by a signature morphism $h : \Sigma \to \Sigma'$ in the sense of [2], Definition 2.4.

Now, by Proposition 3.2 (ii) or Proposition 3.3, the consequence relation $\Vdash_{\mathcal{L}}$ given by an abstract logic $\mathcal{L}$ yields a "logical system" as defined in [2]. Furthermore, any logic map $h : \mathcal{L} \to \mathcal{L}'$ is also a logical system morphism, Definition 2.5 in [2]. On the other hand, a logical system can be transformed into an abstract logic by taking the deductively closed sets as theories. A logical system morphism then turns out to be a logic map, if the assumptions of Proposition 3.3 hold. In the case of normal, L-surjective logic maps (i.e., logic isomorphisms), we are able to prove the following:

Theorem 4.19. *Let $\mathcal{L}$ and $\mathcal{L}'$ be logics (that is, abstract logics obying the above restrictions* (i) *and* (ii)*, or logical systems, depending on the point of view). The following conditions are equivalent:*

- *$\mathcal{L}$ and $\mathcal{L}'$ are isomorphic abstract logics (i.e., there is a logic isomorphism).*
- *$\mathcal{L}$ and $\mathcal{L}'$ are equipollent logical systems and [$\mathcal{L}$ viewed as an abstract logic is singular if and only if $\mathcal{L}'$ viewed as an abstract logic is singular].*

Proof. First, we suppose that there is a logic isomorphism $h : \mathcal{L} \to \mathcal{L}'$ from logic $\mathcal{L}$ to logic $\mathcal{L}'$. By Theorem 4.15, there is a logic isomorphism $h' : \mathcal{L}' \to \mathcal{L}$, such that

(i) $h'\big(h(a)\big) =_{\mathcal{L}} a$, and
(ii) $h(h'(a')) =_{\mathcal{L}'} a'$

holds, for all $a \in Expr_{\mathcal{L}}$ and all $a' \in Expr_{\mathcal{L}'}$. In particular, h and h' are logic maps and therefore logical system morphisms in the sense of [2]. Now it follows immediately from Proposition 4.3 of [2] that $\mathcal{L}$ and $\mathcal{L}'$ are equipollent.

Since $h^{-1}(Expr_{\mathcal{L}'}) = Expr_{\mathcal{L}}$ and $h'^{-1}(Expr_{\mathcal{L}}) = Expr_{\mathcal{L}'}$ and h, h' are logic maps, we get: $\mathcal{L}$ is singular iff $Expr_{\mathcal{L}}$ is a $\mathcal{L}$-theory iff $Expr_{\mathcal{L}'}$ is a $\mathcal{L}'$-theory iff $\mathcal{L}'$ is singular. Hence, (ii) holds.

Now let us assume that $\mathcal{L}$ and $\mathcal{L}'$ are equipollent and [$\mathcal{L}$ is singular if and only if $\mathcal{L}'$ is singular]. By Proposition 4.3 of [2], there are logical system morphisms h and h' satisfying (i) and (ii) above, for all $a \in Expr_{\mathcal{L}}$ and all $a' \in Expr_{\mathcal{L}'}$. It follows that h and h' are L-surjective functions. Let $a \neq_{\mathcal{L}} b$ (a, b are not $\mathcal{L}$-equivalent). We may assume that $a \nVdash_{\mathcal{L}} b$. Then follows that $h(a) \nVdash_{\mathcal{L}'} h(b)$, since h and h' are logical system morphisms and (i) and (ii) hold. Thus, h is also L-injective. The same holds for h'.

We show that h and h' are logic maps. In order to see that h is a logic map, we must prove that for each $T' \in Th_{\mathcal{L}'}$, $h^{-1}(T') = T$ for some $T \in Th_{\mathcal{L}}$. So let $T' \in Th_{\mathcal{L}'}$. In the same way as in the proof of Proposition 3.3 one shows that $h^{-1}(T') =: T$ is deductively closed. If $\mathcal{L}, \mathcal{L}'$ are singular, then T is trivially consistent, thus a theory. So let us suppose that $\mathcal{L}, \mathcal{L}'$ are regular. It remains to prove that T is consistent. Since $Expr_{\mathcal{L}'}$ is not a theory, there is some expression

$b' \in Expr_{\mathcal{L}'}$ such that $T' \nVdash_{\mathcal{L}'} b'$ (choose $b' \in Expr_{\mathcal{L}'} \smallsetminus T'$). Since h is L-surjective, $h(T) \nVdash_{\mathcal{L}'} b'$. Furthermore, there is some $b \in Expr_{\mathcal{L}}$ such that $h(b) =_{\mathcal{L}'} b'$. Thus, $h(T) \nVdash_{\mathcal{L}'} h(b)$. Since h is a logic system morphism (i.e., a uniform translation), $T \nVdash_{\mathcal{L}} b$. Since $\mathcal{L}$ is regular, we conclude that $T = h^{-1}(T')$ is consistent. This shows that h is a logic map.

We show that h is normal. Let $T \in Th_{\mathcal{L}}$ and let T' be the deductive closure of $h(T)$ in $\mathcal{L}'$. Since h is L-injective, $h^{-1}(T') = T$. Thus, it is sufficient to verify that T' is consistent, i.e., a theory. We may assume that $\mathcal{L}, \mathcal{L}'$ are regular, otherwise the assertion would be trivial. Since T is consistent (and $\mathcal{L}$ is regular), there is some expression $b \in Expr_{\mathcal{L}}$ such that $T \nVdash_{\mathcal{L}} b$ (again, choose $b \in Expr_{\mathcal{L}} \smallsetminus T$). From (i) it follows that $h(T) \nVdash_{\mathcal{L}'} h(b)$. Since T' is the deductive closure of $h(T)$, we get $T' \nVdash_{\mathcal{L}'} h(b)$. Since $\mathcal{L}'$ is regular, this means that T' is consistent.

Hence, h is a normal, L-surjective logic map. By Theorem 4.13, h is a logic isomorphism. □

5. Conclusions

In order to establish the fundamentals of a general theory of logics we have based our investigations on a concept of abstract logic which is general enough to be independent of particular model-theoretic or proof-theoretic/syntactical assumptions. On the other hand, it is still strong enough to allow a notion of consistency that coincides with model-theoretic satisfaction and it yields a consequence relation that satisfies the three well-known Tarksi-Axioms of monotonicity, extensiveness and idempotence. We further develop the topological approach of [4] and are able to generalize many results in this broader context. In particular, we develop a general theory of maps between logics and study conditions under which these maps give rise to continuous or open functions between the topological spaces induced by the corresponding logics. We compare our logic maps to the well-known notion of "translation" and discover connections between our approach and the category-theoretic concept of institution. This latter result was already shown in a similar form in the context of model-theoretic abstract logics [4]. It seems that abstract logics may offer a local perspective whereas institutions take a global view on the same things.

We have defined a notion of logic homomorphism as a function on the set of expressions that works in the same way as a continuous map on the respective topological theory space. Logic homomorphisms are logic maps that preserve more topological structure and expressive power of a logic. This fact is illustrated by Example 8. The study of logic homomorphisms culminates in the notion of logic isomorphism, which in this paper is defined in a more flexible and general way as in [4]. A logic isomorphism gives rise to an equivalence relation on the class of abstract logics. We propose this notion as an adequate concept of equivalence of abstract logics: since it induces a homeomorphism, it preserves all topological properties of the theory space. Finally, we show that under certain restrictions our concept of logic isomorphism is equivalent to "equipollence of logical systems" [2]. The comparison of logic isomorphisms to the notion of equipollence is of particular interest: whereas a logic isomorphism is a logic map that induces a homeomorphism of the corresponding topologies, equipollence between two logical systems means that there is some uniform translation between the languages. This translation induces a lattice-isomorphism on the corresponding theory spaces, where a theory space here is viewed as a complete lattice. The paper [2] concentrates on equivalence of logical systems and does not elaborate a general theory of such maps between logics that induce homomorphisms between the respective theory spaces (considered as complete latices). It would be interesting to compare such a lattice-based theory to our approach which is based on general topology.

In order to prove the equivalence of the two concepts "logic isomorphism" and "equipollence" we were forced to adapt our notions of abstract logic and logic maps to the corresponding, more restrictive, notions given in [2]. If we look at these restrictions, in particular the syntactical/category-theoretical conditions and rules of a *uniform* translation (a logical system morphism) of [2], then we may conclude that our notion of equivalence of logics is more flexible and applicable to a broader context: it only depends on the structure of the respective topological theory space.

We believe that the concepts and results developed in this paper will lead to further research on the topological classification of classes of abstract logics. The topological characterization of boolean abstract logics as given above serves here as a first example. The elaboration of a category-theoretic framework with abstract logics as objects and logic maps as morphisms may also be a promising task for future work. Some results of this paper (see Corollary 3.23) may serve as a motivation to study in more detail the relationships between abstract logics and the category-theoretic concept of institutions.

References

[1] J.-Y. Beziau (Ed.); Logica Universalis: Towards a General Theory of Logic; Birkhäuser Verlag, Basel, Switzerland, Second edition 2007.

[2] C. Caleiro, R. Goncalves; Equipollent Logical Systems; in: [1], p. 99–112.

[3] M. E. Coniglio; Towards a Stronger Notion of Translation Between Logics; Manuscrito – Rev. Int. Fil. Campinas, Vol. 28, no. 2, p. 231–262, Jul.–Dez. 2005

[4] S. Lewitzka; A Topological Approach to Universal Logic: Model-Theoretical Abstract Logics; in: [1], p. 35–63.

[5] T. Mossakowski, J. Goguen, R. Diaconescu, A. Tarlecki; What is a Logic?; in: [1], p. 113–134.

[6] S. Pollard; Homeomorphism and the Equivalence of Logical Systems; Notre Dame Journal of Formal Logic, 39, 1998, p. 422–435.

[7] G. Priest; An Introduction to Non-Classical Logic; Cambridge University Press, 2001.

[8] P. Zeitz; Parametrisierte $\in_T$ $-Logik$, Eine Theorie der Erweiterung abstrakter Logiken um die Konzepte Wahrheit, Referenz und klassische Negation. Logos Verlag Berlin, 2000. Dissertation, Technische Universität Berlin, 1999.

Steffen Lewitzka
Federal University of Bahia – UFBA
Instituto de Matemática
Departamento de Ciência da Computação
Laboratório de Sistemas Distribuídos – LaSiD
Av. Ademar de Barros
CEP 40170-110 Salvador da Bahia, BA
Brazil
e-mail: `steffenlewitzka@web.de`

Submitted: 12 January 2006.
Accepted: 17 February 2007.
Published Online First: 8 August 2007.