Multidimensional Rovella-like attractors

V. Araújo [a],[*],[1], A. Castro [b], M.J. Pacifico [a],[1], V. Pinheiro [b]

[a] *Instituto de Matemática, Universidade Federal do Rio de Janeiro, C.P. 68.530, 21.945-970, Rio de Janeiro, RJ, Brazil*
[b] *Departamento de Matemática, Universidade Federal da Bahia, Av. Ademar de Barros s/n, 40170-110 Salvador, Brazil*

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We present a multidimensional flow exhibiting a Rovella-like attractor: a transitive invariant set with a non-Lorenz-like singularity accumulated by regular orbits and a multidimensional non-uniformly expanding invariant direction. Moreover, this attractor has a physical measure with full support and persists along certain sub-manifolds of the space of vector fields. As in the 3-dimensional Rovella-like attractor, this example is not robust. As a sub-product of the construction we obtain a new class of multidimensional non-uniformly expanding endomorphisms without any uniformly expanding direction, which is interesting by itself. Our example is a suspension (with singularities) of this multidimensional endomorphism.

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* Corresponding author.
  E-mail addresses: vitor.araujo@im.ufrj.br (V. Araújo), armandomat@yahoo.com.br (A. Castro), pacifico@im.ufrj.br
(M.J. Pacifico), viltonj@ufba.br (V. Pinheiro).

1. Introduction and statements of the results

Flows in compact (two-dimensional) surfaces are very well understood since the groundbreaking work of Peixoto [25,26]. A theory of three-dimensional flows has steadily been developing since the characterization of robust invariant sets in [21]. The study of singular attractors for flows in higher dimensions is, however, mostly open; see [9,20] and the recent book [7] and references therein.

For non-robust but persistent attractors, like the Rovella attractor presented in [31], there is still no higher-dimensional analogue. Other previous constructions yielding similar behavior can be found in, e.g., [33,32,16,36,12,11]. Of course, one may embed the usual Rovella attractor (also known as *contracting Lorenz* attractor) into flows in any dimension, just by "multiplying by a strong contraction" (the attractor is contained in a three-dimensional sub-manifold, which is invariant and normally contracting for the flow). But this procedure leads to flows without new dynamical phenomena. Moreover, it remains an open problem, since the introduction of the contracting Lorenz models about two decades ago, whether persistent non-robust attractors for singular flows may contain singularities with multidimensional expanding directions.

Here we present a positive solution to this problem.

In a rather natural way, since low-dimensional dynamics is much better understood than the dynamics of general systems, the techniques used in the mathematical analysis of three-dimensional attractors for flows frequently depend on a dimensional reduction through projection along a stable manifold inside some Poincaré cross-section. This method yields a one-dimensional system whose
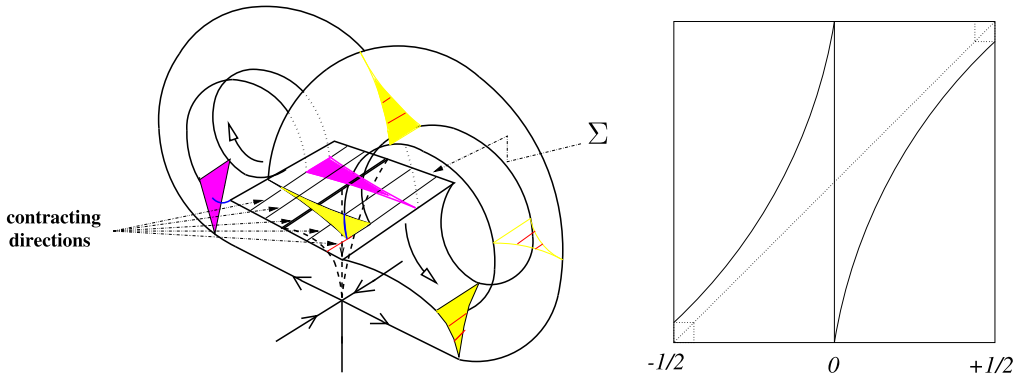
Fig. 1. On the left, the geometric Lorenz attractor with the contracting directions on the cross-section Σ; on the right, the Lorenz one-dimensional transformation.

dynamics can be nearly completely understood as well as the dynamics of its small perturbations; see e.g. [19,10].

In this work we start a rigorous study of a proposed higher-dimensional analogue of the three-dimensional Rovella attractor. Other examples of higher-dimensional chaotic attractors have been recently presented, e.g. by Bonatti, Pumariño and Viana in [9] and by Shil'nikov and Turaev in [35], but these are robust sets, while our construction leads to a persistent non-robust singular attractor.

In [9] the authors define a uniformly expanding map on a higher-dimensional torus, suspend it as a time-one map of a flow, and then singularize the flow adding a singularity in a convenient flow-box. This procedure creates a new dynamics on the torus presenting a multidimensional version of the one-dimensional expanding Lorenz-like map, and a flow with *robust multidimensional Lorenz-like attractors*: the singularity contained in the attractor may have any number of expanding eigenvalues, and *the attractor remains transitive in a whole neighborhood of the initial flow.* The construction in [35] is also robust but yields a quasi-attractor: the attracting invariant set is not transitive but it is the maximal chain recurrence class in its neighborhood.

For the case of the Lorenz attractor and singular-hyperbolic attractors in general, see e.g. [39] and [7]. In this class the equilibrium accumulated by regular orbits inside the attractor has real eigenvalues $\lambda_{ss} < \lambda_s < 0 < \lambda_u$ satisfying $\lambda_s + \lambda_u > 0$ and a type of global (collection of) cross-section(s) can be defined endowed with invariant stable laminations. The quotient of the return map to the global cross-section over the stable leaves (see Fig. 1) is the one-dimensional Lorenz transformation. Our goal is to construct a flow such that "after the identification by the stable directions", the first return map in a certain cross-section M is a multidimensional version of the one-dimensional map of the contracting Lorenz model (or Rovella attractor); see [31].

A Rovella-like attractor is the maximal invariant set of a geometric flow whose construction is very similar to the one that gives the geometric Lorenz attractor [13,1,7], except for the fact that the eigenvalues relation $\lambda_u + \lambda_s > 0$ there is replaced by $\lambda_u + \lambda_s < 0$, where $\lambda_u > 0$ and λ_s is the weakest contractive eigenvalue at the origin. We remark that, unlike the one-dimensional Lorenz map obtained from the usual construction of the geometric Lorenz attractor, a one-dimensional map associated to the contracting Lorenz model has a criticality at the origin, caused by the eigenvalue relation $\lambda_u + \lambda_s < 0$ at the singularity. In Fig. 2 we present some possible one-dimensional maps obtained through quotienting out the stable direction of the return map to the global cross-section of the geometric model of the contracting Lorenz attractor, as in Fig. 1.

The interplay between expansion away from the critical point with visits near the criticality prevents this system to have uniform expansion and also prevents robustness, that is, the attractor is not transitive on a whole neighborhood of the original flow. However the Rovella attractors are *persistent*.

We say that an attractor Λ of a vector field $X \in \mathfrak{X}^3(M)$ is *k-dimensionally persistent*, if there exists a C^∞ sub-manifold $\mathcal{P}$ of $\mathfrak{X}^3(M)$ with codimension k and Λ admits a neighborhood U in M such that for every $Y \in \mathcal{T} := \{Y \in \mathfrak{X}^3(M): \bigcap_{t \geqslant 0} Y^t(U)$ is transitive$\}$ we have
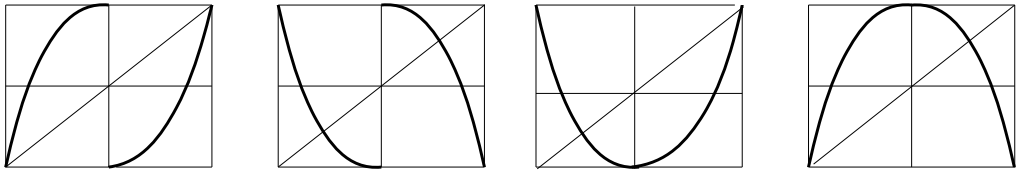
Fig. 2. Several possible cases for the one-dimensional map for the contracting Lorenz model.

$$\lim_{r \to 0} \frac{m(\mathcal{T} \cap S \cap B_r(Y))}{m(S \cap B_r(Y))} = 1$$

where S is any k-dimensional sub-manifold of $\mathfrak{X}^3(M)$ intersecting $\mathcal{P}$ transversely, m is k-dimensional Lebesgue (volume) measure in S and $B_r(Y)$ is the ball of radius $r > 0$ in S in the C^3 topology.

Rovella in [31] showed that this class of three-dimensional attractors is 2-dimensionally persistent in the C^3 topology. That is, *for generic parameterized families of vector fields passing through the original vector field X_0, the parameters corresponding to transitive attractors, with the same features of the Rovella attractor, form a positive Lebesgue measure subset and full density at X_0.* We stress that Rovella-like and Lorenz-like attractors are rather natural dynamical models since they appear in the generic unfolding of resonant double homoclinic loops; see e.g. [28–30,22,23].

In this work we provide a multidimensional counterpart of this result. First we obtain the attractor as follows.

Theorem A. *For any dimension $m = k + 5$, $k \in \mathbb{Z}^+$, there exists a C^∞ vector field $X \in \mathfrak{X}^\infty(M^m)$ on an m-dimensional manifold exhibiting a singular attractor Λ, containing a pair of hyperbolic singularities s_0, s_1 with different indexes in a trapping region U. Moreover*

(1) *there exists a map $R : \Sigma \circlearrowleft$ on a $(k+4)$-dimensional cross-section Σ of the flow of X such that*
    (a) *the set Λ is the suspension of an attractor $\Lambda_\Sigma \subset \Sigma$ with respect to R,*
    (b) *R admits a 3-dimensional stable direction E^s and $(k+1)$-dimensional center-unstable direction E^c such that $E^s \oplus E^c$ is a partially hyperbolic splitting of $T\Sigma \mid \Lambda_\Sigma$,*
    (c) *Λ_Σ supports a physical measure ν for R;*
(2) *Λ is the support of a physical hyperbolic measure μ for the flow X^t: the ergodic basin of μ is a positive Lebesgue measure subset of U and every Lyapunov exponent of μ along the suspension of the bundle E^c is positive, except along the flow direction.*

To prove Theorem A we follow the same strategy of [9] with two main differences.

On the one hand, since we aim at a multidimensional version of the one-dimensional map of the contracting Lorenz model, we have to deal with critical regions, that is, regions where the derivative of the return map to a global cross-section vanishes. Because of this, proving the existence of non-trivial attractors for the flow arising from such construction requires a more careful analysis. Indeed, as in the one-dimensional case, depending on the dynamics of the critical region, every attractor for the return map may be periodic (trivial).

On the other hand, our construction leads to the presence of a pair of hyperbolic saddle equilibria accumulated by regular orbits inside the attractor but *with different indexes* (the dimension of their stable manifolds). This feature creates extra difficulties for the analysis of the possible dynamics arising under small perturbations of the flow.

Typically, when the critical region is non-recurrent (which corresponds to Misiurewicz maps in one-dimensional dynamics), most of the difficulties introduced by the critical region can be bypassed. That is one of the main reasons for us to construct a kind of multidimensional Misiurewicz dynamics. In general, such critical regions in dimension greater than one are sub-manifolds, and one cannot rule out that they intersect each other under the action of the dynamics. Albeit this, we shall exhibit a class of multidimensional Misiurewicz-like endomorphisms that appears naturally in a flow dynamics;

see Theorem 2.1 in Section 2. This is an example of non-uniformly expanding dynamics in higher dimensions which *does not exhibit any uniformly expanding direction* and is conjugated to a skew-product of a quadratic map with an expanding map, with the exception of at most two orbits.

This is the basic dynamics which we modify to obtain the return map R to a cross-section of the flow, exhibiting an expanding invariant torus $\mathbb{T}_1^k$ that will absorb the image of the critical region after the singularization of the associated flow. *By topological reasons, this map cannot be seen as a time-one map of a suspension flow*: *locally its degree is not constant.* To bypass this new difficulty, we realize this map as a first return map of a flow *with singularities* (after identification by stable directions). Afterwards, we singularize a periodic orbit of this flow, i.e., we introduce a new singularity s of saddle-type, with $(k+1)$-dimensional unstable manifold and 4-dimensional stable manifold. Moreover, all the eigenvalues of s are real and if $\sigma_{s,i}$ and $\sigma_{u,j}$ denote the stable and the unstable eigenvalues at s respectively, then $\max\{\sigma_{s,i}\} + \max\{\sigma_{u,j}\} < 0$ for $1 \leqslant i \leqslant 4$ and $1 \leqslant j \leqslant k+1$. We say that this kind of singularity is a *Rovella-like singularity*. The resulting flow will present a multidimensional transitive *Rovella-like attractor*, supporting a physical measure, as stated in Theorem A.

The existence of the physical/SRB measure for the original flow is obtained taking advantage of the fact that, through identification of stable leaves, we can project the dynamics of the first return map R of the flow to a global cross-section into a one-dimensional transformation with a Misiurewicz critical point.

We point out that *the analysis of the dynamics of most perturbations of our flow cannot be easily reduced* (*perhaps not at all*) *to a one-dimensional model.* This indicates that intrinsic multidimensional tools should be developed to fully understand this class of flows. Thus extra difficulties arise to verify that this kind of multidimensional contracting Lorenz attractor is persistent. We obtain the following partial result in this direction.

Theorem B. *For any $k \in \mathbb{Z}^+$, there exists a $(k+2)$-codimension sub-manifold $\mathcal{P}$ of the space of C^2 vector fields $\mathfrak{X}^2(M)$ such that*

(1) *the vector field X from Theorem A belongs to $\mathcal{P}$ and, for all $Y \in \mathcal{P}$ in a neighborhood $\mathcal{U}$ of X in $\mathfrak{X}^2(M)$, the Poincaré return map R_Y to the cross-section Σ admits a 3-dimensional strongly contracting C^γ foliation $\mathcal{F}$, for some $\gamma > 1$. The induced map g_Y on the quotient of Σ over $\mathcal{F}$ is a C^γ endomorphism on a cylinder $[-1,1] \times \mathbb{T}^k$;*
(2) *for vector fields $Y \in \mathcal{U} \setminus \mathcal{P}$, the Poincaré return map R_Y to the cross-section Σ admits a one-dimensional strongly contracting C^γ foliation $\mathcal{F}$, for some $\gamma > 1$. The induced map g_Y on the quotient of Σ over $\mathcal{F}$ is a C^γ endomorphism on a manifold diffeomorphic to the unit ball in $\mathbb{R}^{k+3}$;*
(3) *for a vector field $Y \in \mathcal{U} \cap \mathcal{P}$, if the quotient map g_Y sends the critical set inside the stable manifold of the sink $p(Y)$, then this stable manifold contains the trapping region U except for a zero Lebesgue measure subset.*

Next we give the precise definitions and concepts involved in the previous statements.

1.1. Preliminary definitions and conjectures

In what follows M is a compact boundaryless finite-dimensional manifold and $\mathfrak{X}^1(M)$ is the set of C^1 vector fields on M, endowed with the C^1 topology. From now on we fix some smooth Riemannian structure on M and an induced normalized volume form m that we call Lebesgue measure. We write also dist for the induced distance on M and $\|\cdot\|$ for the induced Riemannian norm on TM. Given $X \in \mathfrak{X}^1(M)$, we denote by X^t, $t \in \mathbb{R}$, the flow induced by X, and if $x \in M$ and $[a,b] \subset \mathbb{R}$ then $X^{[a,b]}(x) = \{X^t(x), \ a \leqslant t \leqslant b\}$.

We say that a differentiable map $f : M \circlearrowleft$ is C^{1+} if the derivative Df of f is *Hölder*: there are $\alpha, C > 0$ such that for every $x \in M$ we can find parametrized neighborhoods $U = \phi(U_0)$ of x and $V = \psi(V_0)$ of $f(x)$ in M, where U_0, V_0 are neighborhoods of 0 in $\mathbb{R}^{\dim(M)}$ and ϕ, ψ are parametrizations of M, such that for all $y_1, y_2 \in U$

$$\left\| D\left(\psi^{-1} \circ f \circ \phi\right)(y_1) - D\left(\psi^{-1} \circ f \circ \phi\right)(y_2) \right\| \leqslant C \operatorname{dist}(y_1, y_2)^\alpha.$$

A point $p \in M$ is a *periodic point* for X^t if $X(p) \neq 0$ and there exists $\tau > 0$ such that $X^\tau(p) = p$. The minimal value of τ such that $X^\tau(p) = p$ is the *period* of p. If p is a periodic point, we also say that the orbit $\mathcal{O}(p) = \{X^t(p): t \in \mathbb{R}\}$ of p is a *periodic orbit*. A *singularity* σ is an equilibrium point of X^t, that is, $X(\sigma) = 0$. If $X(p) \neq 0$ then we say that p is a *regular point* and its orbit $\mathcal{O}(p)$ is a *regular orbit*.

Let Λ be a compact invariant set of $X \in \mathfrak{X}^1(M)$. We say that Λ is an *attracting set* if there exists a *trapping region*, i.e. an open set $U \supset \Lambda$ such that $\overline{X^t(U)} \subset U$ for $t > 0$ and $\Lambda = \bigcap_{t \in \mathbb{R}} X^t(U)$. Here $\overline{A}$ means the topological closure of the set A in the manifold we are considering.

We say that an attracting set Λ is *transitive* if it is equal to the ω-limit set of a regular X-orbit. We recall that the ω-limit set of a given point x with respect to the flow X^t of X is the set $\omega(x)$ of accumulation points of $(X^t(x))_{t>0}$ when $t \to +\infty$. An *attractor* is a transitive attracting set and a *singular attractor* is an attractor which contains some equilibrium point of the flow. An attractor is *proper* if it is not the whole manifold. An invariant set of X is *non-trivial* if it is neither a periodic orbit nor a singularity.

Definition 1.1. Let Λ be a compact invariant set of a C^{1+} map $f: M \circlearrowleft$, $c > 0$, and $0 < \lambda < 1$. We say that Λ has a (c, λ)-*dominated splitting* if the bundle over Λ splits as a Df-invariant sum of sub-bundles $T_\Lambda M = E^s \oplus E^{cu}$, such that for all $n \in \mathbb{Z}^+$ and each $x \in \Lambda$

$$\left\| Df^n \mid E_x^s \right\| \cdot \left\| \left(Df^n \mid E_x^{cu} \right)^{-1} \right\| < c\lambda^n. \tag{1}$$

We say that an f-invariant subset Λ of M is *partially hyperbolic* if it has a (c, λ)-dominated splitting, for some $c > 0$ and $\lambda \in (0, 1)$, such that the sub-bundle E^s is uniformly contracting: for all $n \in \mathbb{Z}^+$ and each $x \in \Lambda$ we have

$$\left\| Df^n \mid E_x^s \right\| < c\lambda^n. \tag{2}$$

We denote by $\overline{A}$ the topological closure of the set $A \subset M$ in what follows.

Definition 1.2. A C^{1+} map $g: M \to M$ is *non-uniformly expanding* if there exists a constant $c > 0$ such that

$$\liminf_{n \to +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \left\| Dg\left(g^j(x)\right)^{-1} \right\| \leqslant -c < 0 \quad \text{for Lebesgue almost every } x \in M.$$

Non-uniform expansion ensures the existence of absolutely continuous invariant probability measures under some mild extra assumptions on g, see below. One property of such measures is that they have a "large ergodic basin".

Definition 1.3 (*Physical measure and ergodic basin for flows*). We say that an invariant probability measure μ is a *physical measure* for:

- the flow given by a field $X: M \to TM$ if there exists a positive Lebesgue measure $B(\mu) \subset M$ such that for all $x \in B(\mu)$ we have $\frac{1}{T} \int_0^T \varphi \circ X^t(x)\, dt \xrightarrow[T \to +\infty]{} \int_M \varphi\, d\mu$ for all continuous functions $\varphi: M \to \mathbb{R}$;
- the map $g: M \to M$ if there exists a positive Lebesgue measure $B(\mu) \subset M$ such that for each $x \in B(\mu)$ we have $\frac{1}{n} \sum_{j=0}^{n-1} \varphi(g^j(x)) \xrightarrow[n \to +\infty]{} \int_M \varphi\, d\mu$ for all continuous functions $\varphi: M \to \mathbb{R}$.

The set $B(\mu)$ is called the *ergodic basin* of μ.

Note that Theorems A and B show that the singular attractor Λ is not robustly transitive not even robust: there exist arbitrary small perturbations Y of the vector field X such that the orbits of the flow of Y, in a full Lebesgue measure set of points in U, converge to a periodic attractor (a periodic sink for the flow). The singular attractor Λ is not partially hyperbolic in the usual sense (1) and (2) adapted to the flow setting, since we can only define the splitting on the points of Λ which do not converge to the singular points s_0, s_1, that is, we exclude the stable set of the singularities within Λ. The remaining set, however, has full measure with respect to μ. Since the equilibria s_0, s_1 contained in Λ are hyperbolic with different indexes (i.e. the dimension of their stable manifolds) we believe the following can be proved.

Conjecture 1. *It is not possible to extend the dominated splitting on Λ away from equilibria to the equilibria s_0, s_1 which belong to Λ.*

In addition, the attractor Λ for X in U is such that the Jacobian along any 2-plane P_x, inside the central sub-bundle E_x^c for Lebesgue almost all $x \in U$, is *asymptotically expanded but not uniformly expanded*, that is we can find a constant $c > 0$ such that $\lim_{t \to +\infty} \frac{1}{t} \log |\det DX^t \mid P_x| \geqslant c$, but $\frac{1}{t} \log |\det DX^t \mid P_x|$ can take an arbitrary long time (depending on x) to become positive.

For the remaining cases, not analyzed in Theorem B, we conjecture that the quotient map behaves in a similar way to the typical perturbations of a smooth one-dimensional multimodal map.

Conjecture 2. *For a vector field $Y \in \mathcal{U}$, where $\mathcal{U}$ is the neighborhood of X in $\mathfrak{X}^2(M)$ introduced in Theorem B, if the quotient map g_Y does not send the critical set inside the stable manifold of the sink $p(Y)$, then the complement of this stable manifold contains the basin of a physical measure for g_Y whose Lyapunov exponents are positive.*

To make any progress in the understanding of this conjecture one needs to study the interplay between the critical set, the expanding behavior in some regions of the space, and the stable manifold of the sink, in a higher-dimensional setting. We believe this will demand the development of new ideas in dynamics and ergodic theory.

1.2. Organization of the text

We present the construction of the vector field X in stages in Section 2. We start by constructing a non-uniformly expanding higher-dimensional endomorphism with critical points in Section 2.1. This provides a starting point for the Poincaré return maps $R : \Sigma \circlearrowleft$ of the statement of Theorem A. Then we adapt this first construction to become the quotient of the return map of the flow we will construct, in Section 2.2. In Section 2.3 we start the construction of the singular flow we are interested in. This is done again in stages, and here we obtain a first candidate. Next we obtain the vector field X after perturbing the candidate in Section 2.4.

We study the properties of X and its unfolding in Section 3. We prove the existence of the dominated splitting for the return map to the cross-section and describe the construction of the physical measure with positive multidimensional Lyapunov exponents in Section 3.1, completing the proof of Theorem A. The details on the existence of the physical measure for the return map are left for Section 4, where "Benedicks–Carleson type" arguments are adapted to our higher-dimensional setting.

The unfolding of the vector field X is studied in Section 3.2, where items (1)–(2) of Theorem B are proved and the argument for the proof of item (3) is described. The proof of this last item is given in Section 5.

We end with two appendixes. Appendix A provides an adaptation of a major technical tool to our setting to prove the existence of an absolutely continuous measure for higher-dimensional non-uniformly expanding maps. Appendix B provides a topological argument for the existence of a certain isotopy we need during the construction of the vector field X.
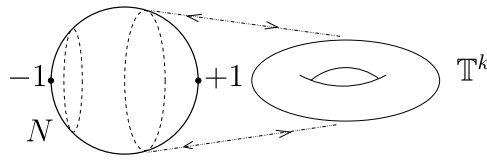
Fig. 3. The "torusphere": each parallel is a k-torus.
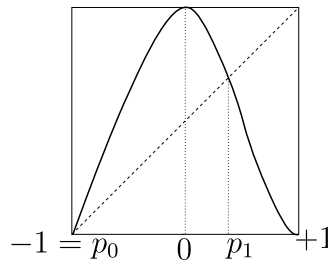
Fig. 4. The one-dimensional map g_0.

2. The construction

Here we prove Theorem A. We start by providing an example of non-uniformly expanding dynamics in higher dimensions which is interesting by itself since it *does not exhibit any uniformly expanding direction*. Then we adapt this example to obtain the Poincaré return map R in the statement of Theorem A. At the end of this section, we show how this yields a construction of a singular attractor with $k+1$ non-uniformly expanding directions on a compact boundaryless $(k+5)$-dimensional manifold M.

2.1. An example of non-uniformly expanding dynamics in high dimension

We start by defining a non-uniformly expanding endomorphism of a $(k + 1)$-dimensional manifold N.

Let $\Upsilon : \mathbb{R} \times \mathbb{C}^k \to \mathbb{R} \times \mathbb{C}^k$ be given by $(t, z) \mapsto (\cos(\pi t), z \sin(\pi t))$. We consider $\mathbb{T}^k = \mathbb{S}^1 \times \overset{k}{\cdots} \times \mathbb{S}^1$ where $\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$ and let $N = \Upsilon([-1, 1] \times \mathbb{T}^k)$. Clearly N has a natural differential structure: $N = G^{-1}(\{1, \ldots, 1\})$ for

$$G : \mathbb{R} \times \mathbb{C}^k \to \mathbb{R}^k : (t, z_1, \ldots, z_k) = \left(t^2 + |z_1|^2, \ldots, t^2 + |z_k|^2\right)$$

which we name "torusphere" and is a manifold of dimension $k + 1$, see Fig. 3.

We remark that this manifold is the boundary of $M := G^{-1}([0, 1]^k)$, which is a "solid torusphere", that is $M \cap (\{t\} \times \mathbb{R}^k) \simeq \mathbb{T}^k \times \mathbb{D}$ for all $-1 < t < 1$, where $\mathbb{D}$ is the unit disk in $\mathbb{C}$. In what follows we write $I = [-1, 1]$.

Let $g_0 : I \circlearrowleft$ be a C^{1+} non-flat unimodal map with the critical points $c_0 = 0$ and $c_1 = g_0(c_0) = 1$ as follows

$$g_0(x) = \begin{cases} \varsigma^+ (|\frac{x}{2}(1 - \frac{x}{2})|^\alpha) & \text{if } x \in [0, 1], \\ \varsigma^- (|x|^\alpha) & \text{if } x \in [-1, 0); \end{cases}$$

for C^∞ diffeomorphisms $\varsigma^\pm : [0, 1] \to I$ such that both ς^+ and ς^- are monotonous decreasing. Moreover we assume that *the critical order α is strictly between 1 and 2*, $1 < \alpha < 2$, and that g_0 satisfies (see Fig. 4):

(1) $g_0(\pm 1) = -1$ and g_0 has exactly two fixed points, namely, $p_0 = -1$ and $0 < p_1 < 1$;
(2) $g_0'(p_0) > 1$ and $g_0'(p_1) < -1$.

It is well known that these maps are non-uniformly expanding and conjugated to the tent map (see the proof of Lemma 2.3 in what follows). Thus, in particular, they are topologically transitive and admit a unique absolutely continuous probability measure supported on the entire interval.

Now we define $f_1 : \mathbb{C}^k \to \mathbb{C}^k$ by $(w_1, \ldots, w_k) \mapsto (w_1^2, \ldots, w_k^2)$. Then the map $g_0 \times f_1 : [-1, 1] \times \mathbb{T}^k \circlearrowleft$ induces a $C^{1+\alpha}$ map $g : N \circlearrowleft$ by $g := \Upsilon \circ (\hat{g}_0 \times f_1) \circ \Upsilon^{-1}$. It is easy to see that g takes meridians onto meridians and parallels onto parallels of the torusphere. This means that the derivative of g preserves the directions associated to such foliations.

Next we state and prove the main properties of the map g defined above. This is the main result of this subsection.

Theorem 2.1. *For any $k \in \mathbb{Z}^+$, g is a non-uniformly expanding map of class C^{1+} on the $(k+1)$-manifold N, without any uniformly expanding directions, admitting an absolutely continuous invariant probability measure with full ergodic basin in N, whose Lyapunov exponents are all positive.*

For the proof of theorem above we start with the following

Claim 2.2. *The map g is a non-uniformly expanding map.*

Using polar coordinates, we can take a parametrization $h : (-1, 1) \times (0, 2\pi)^k \to N$ given by $h(t, \Theta) = h(t, \theta_1, \ldots, \theta_k) := (\sin(\frac{\pi}{2}t), z(\Theta)\cos(\frac{\pi}{2}t))$, where $\Theta := (\theta_1, \ldots, \theta_k)$ and

$$z(\Theta) := \big(\cos(\theta_1), \sin(\theta_1), \ldots, \cos(\theta_k), \sin(\theta_k) \big) \in \mathbb{R}^{2k}.$$

The image of h covers N except for a null Lebesgue measure set. Therefore, the expression of g in these coordinates is

$$\underbrace{\left(\sin\left(\frac{\pi}{2}t\right), \cos\left(\frac{\pi}{2}t\right)z(\Theta) \right)}_{:=x} \mapsto \left(\sin\left(\frac{\pi}{2}g_0(t)\right), \cos\left(\frac{\pi}{2}g_0(t)\right)z(2\Theta) \right).$$

In the meridian directions, this implies that the derivative $Dg(x) : T_x N \to T_{g(x)} N$ takes the vector $v := (\cos(\frac{\pi}{2}t), \sin(\frac{\pi}{2}t)z(\Theta))$ to $Dg(x) \cdot v = g_0'(t) \cdot (\cos(\frac{\pi}{2}g_0(t)), \sin(\frac{\pi}{2}g_0(t))z(2\Theta))$.

We adopt in $\mathbb{R} \times \mathbb{R}^{2k}$ (where N is embedded) the norm $\|(t, z)\| := \sqrt{|t|^2 + \|z\|^2/k}$, where $\|z\|$ is the standard Euclidean norm in $\mathbb{R}^{2k}$. Therefore

$$\frac{\|Dg(x) \cdot v\|}{\|v\|} = |g_0'(t)| \cdot \sqrt{\frac{\cos^2(\frac{\pi}{2}g_0(t)) + \sin^2(\frac{\pi}{2}g_0(t)) \cdot \|z(2\Theta)\|^2/k}{\cos^2(\frac{\pi}{2}t) + \sin^2(\frac{\pi}{2}t) \cdot \|z(\Theta)\|^2/k}}$$

$$= |g_0'(t)| \cdot \sqrt{\frac{\cos^2(\frac{\pi}{2}g_0(t)) + \sin^2(\frac{\pi}{2}g_0(t))}{\cos^2(\frac{\pi}{2}t) + \sin^2(\frac{\pi}{2}t)}} = |g_0'(t)|.$$

Along the directions of the parallels, given any $j \in \{1, \ldots, k\}$ the derivative $Dg(x)$ takes the vector $v_j := (0, \ldots, -\cos(\frac{\pi}{2}t)\sin(\theta_j), \cos(\frac{\pi}{2}t)\cos(\theta_j), 0, \ldots, 0)$ to the vector

$$Dg(x) \cdot v_j = \left(0, \ldots, -2\cos\left(\frac{\pi}{2}g_0(t)\right)\sin(2\theta_j), 2\cos\left(\frac{\pi}{2}g_0(t)\right)\cos(2\theta_j), 0, \ldots, 0 \right).$$

Therefore

$$\frac{\||Dg(x) \cdot v_j\||}{\||v_j\||} = 2 \frac{|\cos(\frac{\pi}{2} g_0(t))| \cdot \sqrt{\sin^2(2\theta_j) + \cos^2(2\theta_j)}}{|\cos(\frac{\pi}{2} t)| \cdot \sqrt{\sin^2(\theta_j) + \cos^2(\theta_j)}} = 2 \left| \frac{\cos(\frac{\pi}{2} g_0(t))}{\cos(\frac{\pi}{2} t)} \right|.$$

We note that for $w = \sum_{j=1}^{k} \alpha_j v_j$, the relation $\frac{\||Dg(x) \cdot w\||}{\||w\||} = 2 |\frac{\cos(\frac{\pi}{2} g_0(t))}{\cos(\frac{\pi}{2} t)}|$ also holds. Let us call E_x the space generated by the directions of the parallels through x in $T_x N$, and $m(x) := \inf_{w \in E_x} \frac{\||Dg(x) \cdot w\||}{\||w\||}$ the minimum norm (or conorm) of $Dg(x)$ restricted to E_x. We have

$$\frac{1}{n} \sum_{j=0}^{n-1} \log m\big(g^j(x)\big) = \log(2) + \frac{1}{n} \sum_{j=0}^{n-1} \log \left| \frac{\cos(\frac{\pi}{2} g_0^{j+1}(t))}{\cos(\frac{\pi}{2} g_0^j(t))} \right| = \log(2) + \frac{1}{n} \log \left| \frac{\cos(\frac{\pi}{2} g_0^n(t))}{\cos(\frac{\pi}{2} t)} \right|.$$

Thus we will have $n^{-1} \sum_{j=0}^{n-1} \log m(g^j(x)) \geqslant \log 2$ whenever $|\cos(\frac{\pi}{2} g_0^n(t))| \geqslant |\cos(\frac{\pi}{2} t)|$, which is true if, and only if, $|g_0^n(t)| \leqslant |t|$.

To conclude the proof of Claim 2.2 we use Lemma 2.3 below, whose proof we postpone to the end of this subsection.

Lemma 2.3. *Given a neighborhood U of c_0, Lebesgue almost every orbit visits U infinitely often.*

Since $c_0 = 0$, Lemma 2.3 ensures that for every given $t \in I \setminus \{0\}$ the inequality $|g_0^n(t)| \leqslant |t|$ is true for infinitely many values of $n \geqslant 1$. This implies that

$$\liminf_{n \to +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \||(Dg \mid E_{g^j(x)})^{-1}\|| \leqslant -\log 2 < 0 \quad \text{for Lebesgue almost every } x. \tag{3}$$

Denoting F_x the direction of the meridian at $T_x N$, for $x = (t, \Theta)$, we also showed that

$$\liminf_{n \to +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \||(Dg \mid F_{g^j(x)})^{-1}\|| = -\limsup_{n \to +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log |g_0'(g_0^j(t))| < 0 \tag{4}$$

for Lebesgue almost every $t \in I$, which is strictly negative by known results on unimodal maps (see e.g. [10]). From (3) and (4) we obtain

$$\limsup_{n \to +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \||Dg(g^j(x))^{-1}\||^{-1} > 0 \quad \text{for Lebesgue almost every } x. \tag{5}$$

Hence, according to [27], g is a non-uniformly expanding map since, besides (5), the orbit of the critical set clearly *does not accumulate* the critical set. This ensures that g admits an absolutely continuous invariant probability measure, which is unique because g is a transitive map (see the proof of Lemma 2.3 below).

To complete the proof of Theorem 2.1, we are left to prove Lemma 2.3.

Proof of Lemma 2.3. It is not difficult to see that g_0 is topologically conjugated to the tent map $T(x) := 1 - 2|x|$ for $x \in I$ under some homeomorphism h of the interval I. Indeed, searching for h of the form $h(x) = x + u(x)$ on each interval $[-1, 0]$ and $[0, 1]$ for some continuous $u : \pm[0, 1] \to I$ with small C^0-norm, we get the relation

$$h\big(g_0(x)\big) = 1 - 2\big|h(x)\big| \quad \text{or} \quad g_0(x) + u\big(g_0(x)\big) = 1 - 2\big|x + u(x)\big|.$$

For $x \in [0, 1]$ we have $h(x) \geqslant 0$ and so we obtain

$$\underbrace{2^{-1}u\big(g_0(x)\big) + u(x)}_{\mathcal{L}(u)x} = \frac{1}{2}\big(1 - 2x - g_0(x)\big).$$

We remark that from this relation it follows that $u(0) = u(1) = 0$ and, moreover, the right-hand side is strictly smaller than $1/2$ uniformly on $[0, 1]$, that is, $\mathcal{L}(u) \leqslant 1/2 - \xi$ for some $0 < \xi < 1/2$. Clearly $\mathcal{L}(u) = \mathcal{I} + L$ where $\mathcal{I}$ is the identity on $C^0([0, 1], I)$ and $(Lu)x = 2^{-1}u(g_0(x))$ has C^0-norm $\leqslant 1/2$. Thus the linear operator $\mathcal{L} : C^0([0, 1], I) \to C^0([0, 1], I)$ admits an inverse. Analogously for the conjugation equation on $[-1, 0]$. So we can find $h = \mathcal{I} + u$ with u having C^0-norm < 1, as needed to ensure that h is invertible, thus a homeomorphism of I.

This guarantees that g_0 is transitive and, in particular, has no attracting periodic orbits.

The subset $K := \bigcap_{n \geqslant 0} T^{-n}([-1, 1 - \varepsilon])$ is a T-invariant Cantor set with zero Lebesgue measure, for each $0 < \varepsilon < 1$. Thus $K_0 := h(K)$ is also a g_0-invariant Cantor set such that, for some $\delta > 0$ and every $z \in K_0$ satisfies $g_0^n(z) \in I \setminus ((-\delta, \delta) \cup (1 - \delta, 1])$ for all $n \geqslant 0$.

That is, K_0 is the set of points whose future orbits under g_0 do not visit a neighborhood V_0 of the critical set, so that $g_0 \mid (I \setminus \overline{V_0})$ is a C^∞ map acting on K_0, by the definition of g_0. Hence in K_0 we have no critical points and no attracting periodic orbits, thus the restriction $g_0 \mid K_0 : K_0 \to K_0$ is a uniformly expanding local diffeomorphism by Mañé's results in [17].

Therefore the Lebesgue measure of K_0 is zero, for otherwise this set would have a Lebesgue density point p. Since g_0 is C^∞ in a neighborhood of K_0, this would imply that K_0 would contain some interval J (see e.g. [38] or [5]). But g_0 is uniformly expanding on K_0, thus the length of the successive images $g_0^k(J)$ of J would grow to at least the length of one of the domains of monotonicity of g_0, in a finite number of iterates. The next iterate would contain the critical point, contradicting the definition of K_0.

It follows that the set $E(\delta)$ of points of I which do not visit a δ-neighborhood of c_0 under the action of g_0 has zero Lebesgue measure, for all small $\delta > 0$. This ensures that $\bigcup_{n > N} \bigcap_{k > n} g_0^{-k}(E(k^{-1}))$ has zero volume for every big $N > 1$. Consequently the set $\bigcap_{n > N} \bigcup_{k > n} g_0^{-k}(M \setminus E(k^{-1}))$ has full measure, and for points in this set there are infinitely many iterates visiting any given neighborhood of c_0. $\square$

This completes the proof of Theorem 2.1.

2.2. The unperturbed basic dynamics

We now adapt the example in Section 2.1 in order to obtain a map f which will be a kind of Poincaré return map for a singular flow, that we will perturb later to obtain a Rovella-like flow. We again construct a map f in the torusphere by defining its action in the meridians and parallels.

Let $f_0 : I \to I$ be a C^{1+} non-flat unimodal map with the critical point $c = 0$, that is,

$$f_0(x) = \begin{cases} \psi^+(x^\alpha) & \text{if } x \in (0, 1], \\ \psi^-(|x|^\alpha) & \text{if } x \in [-1, 0); \end{cases}$$

for smooth monotonous increasing diffeomorphisms $\psi^\pm : [0, 1] \to I$. Moreover we assume that the critical order α is at least 2 and that f satisfies (see Fig. 5):

(1) $f_0(\pm 1) = -1$, f_0 has exactly three fixed points: $p_0 = -1 < p_1 < c = 0 < p_2 < 1$;
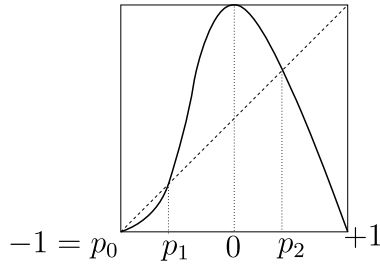(2) $f_0'(p_2) < -1 < 0 \leqslant f_0'(-1) < 1 < f_0'(p_1)$.
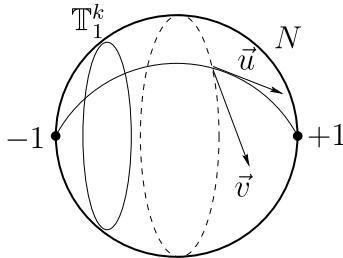
Fig. 5. The one-dimensional map f_0.

Fig. 6. The parallel corresponding to the expanding fixed point is a uniform repeller for f, a vector $\vec{u}$ tangent to a meridian, and a vector $\vec{v}$ tangent to a parallel.

The map $f_0 \times f_1$ induces a $C^{1+\alpha}$ map $f : N \to N$ by $f = \Upsilon \circ (f_0 \times f_1) \circ \Upsilon^{-1}$. Let $\mathbb{T}_1^k = \Upsilon(\{p_1\} \times \mathbb{T}^k)$. Note that $f(\mathbb{T}_1^k) = \mathbb{T}_1^k$, in other words, $\mathbb{T}_1^k$ is positively invariant by f and a uniform repeller, see Fig. 6.

2.3. The unperturbed singular flow

Here we build a geometric model for a $(k + 5)$-dimensional flow X_0^t, $t \geqslant 0$. We write B^n for the n-dimensional unit ball in $\mathbb{R}^n$, that is $B^n := \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : \sum_{i=1}^n x_i^2 < 1\}$.

Recall that $f_1 : \mathbb{T}^k \to \mathbb{T}^k$ is the expanding map defined in Section 2.1. Such map has a lift to an inverse limit which is a higher-dimensional version of a Smale solenoid map, see [34].

More precisely, as shown in Appendix B, given a solid k-torus $\mathcal{T} := \mathbb{T}^k \times \mathbb{D}$, where $\mathbb{D} = \{z \in \mathbb{C}: |z| \leqslant 1\}$, there exists a map $S : e(\mathcal{T}) \to e(\mathcal{T})$ defined on the image of a smooth embedding $e : \mathcal{T} \to B^{k+2}$ such that $\pi_{\mathbb{D}} \circ S = f_1 \circ \pi_{\mathbb{D}}$, where $\pi_{\mathbb{D}} : e(\mathcal{T}) \to e(\mathbb{T}^k \times 0) \simeq \mathbb{T}^k$ is the projection along the leaves of the foliation $\mathcal{F}^s := \{e(\Theta \times \mathbb{D})\}_{\Theta \in \mathbb{T}^k}$ of $\mathcal{T}$. Moreover, F contracts the disks in $\mathcal{F}^s$ by a uniform contraction rate $\lambda \in (0, 1)$. We remark that $\mathbb{T}^k$ is the quotient of $e(\mathcal{T})$ over $\mathcal{F}^s$. From now on we identify $e(\mathcal{T})$ with $\mathcal{T}$ and $\mathbb{T}^k$ with $e(\mathbb{T}^k \times 0)$. The map S is the higher-dimensional version of a Smale solenoid map we mentioned above.

We also have (see Appendix B) that there exists a smooth isotopy $\phi_t : B^{k+2} \to B^{k+2}$ between $S = \phi_1$ and the identity map ϕ_0 on B^{k+2}.

We define a flow between two cross-sections $\Sigma_j = \pi_1(I^2 \times \{3 - j\} \times B^{k+2})$, $j = 1, 2$, by

$$Y^t\big(\pi_1(x_1, x_2, 2, W)\big) = \pi_1\big(x_1, x_2, 2 - t, \phi_t(W)\big), \quad 0 \leqslant t \leqslant 1,$$

where $(x_1, x_2, 2 - t, W) \in I^2 \times [1, 2] \times B^{k+2}$ and

$$\pi_1 : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times B^{k+2} \circlearrowleft, \qquad (x_1, x_2, x_3, W) \mapsto \big(x_1, x_2, x_3, (1 - x_1^2)^{1/2} W\big).$$
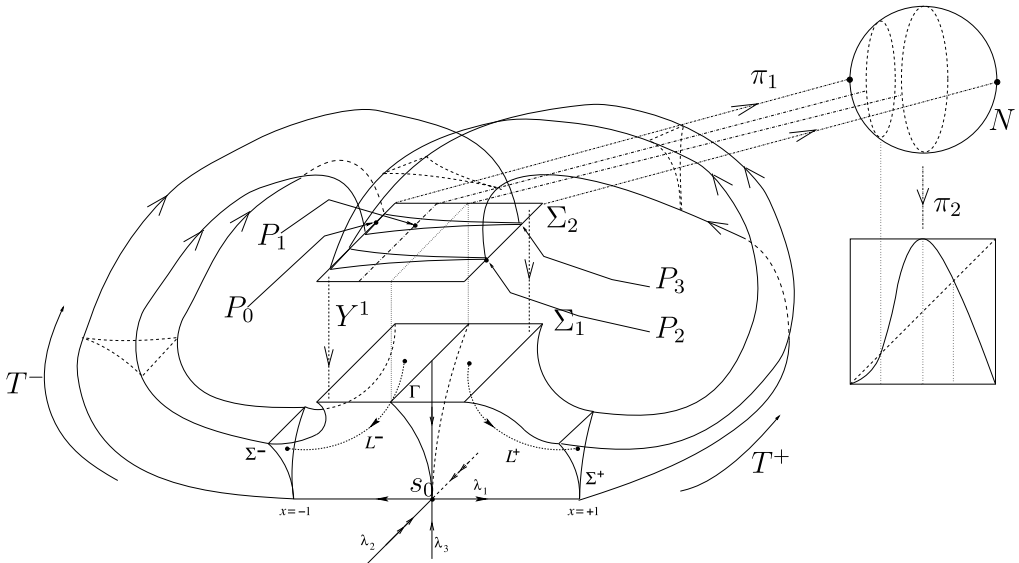
Fig. 7. The starting singular flow with the transformations Y^1, $L^\pm$ and the projections π_1 and π_2.

We note that $\Sigma_j \simeq I \times \overline{B^{k+3}}$ naturally for any $j = 1, 2$. We extend this flow so that there exists a Poincaré return map from Σ_2 to Σ_1 with the properties we need. For this we take a linear flow on $\mathbb{R}^3 \times \mathbb{R}^{k+2}$ with a singularity s_0 at the origin having real eigenvalues $\lambda_1 > 0$, and $\lambda_1 + \lambda_j < 0$ for $2 \leqslant j \leqslant k + 5$. For simplicity, we also assume that $\lambda_j = \lambda_3$ for all $j \geqslant 4$ and that $\alpha := -\lambda_3/\lambda_1$ and $\beta := -\lambda_2/\lambda_1$ satisfy $\beta > \alpha + 2$. This last *strong dissipative condition* on the saddle s_0 ensures that the foliation corresponding to the x_2 direction is dominated, and so persists for all C^2 nearby flows.

We note that the subspace $\{0\} \times \mathbb{R}^2 \times B^{k+2}$, excluded from further considerations, is contained in the stable manifold of s_0 and so its points never return no Σ_j.

We write $(x_1, x_2, 1, W)$ for a point on Σ_2, with $x_i \in \mathbb{R}$ and $W \in B^{k+2}$, and consider the cross-sections $\Sigma^\pm = \{(\pm 1, x_2, x_3, W) : (x_2, x_3) \in I^2, \ W \in \mathbb{R}^{k+2}\}$ to the flow and the Poincaré first entry transformations given by (see Fig. 7):

$$L^\pm : \Sigma_1 \cap \{\mp x_1 > 0\} \to \Sigma^\pm : (x_1, x_2, 1, W) \mapsto \big(\pm 1, x_2 |x_1|^\beta, |x_1|^\alpha, |x_1|^\alpha W\big). \tag{6}$$

Remark 2.4. The maps $L^\pm$ given in (6) are clearly Hölder maps in their domain of definition. Moreover the time the flow needs to take the point $(x_1, x_2, 1, W)$ to $\Sigma^\pm$ is given by $-\log |x_1|$, where $|x_1|$ is the distance to the local stable manifold of s_0 on Σ_2.

Now we define diffeomorphisms from the image of $L^\pm$ to Σ_1 which can be realized as the first entry maps from $\Sigma^\pm$ to Σ_1 under a flow defined away from the origin in $\mathbb{R}^{k+5}$. Since we want to define a flow with an attractor containing the origin in $\mathbb{R}^{k+5}$, we need to ensure that the return map defined on Σ_1 through the composition of all the above transformations does preserve the family of tori together with their stable foliations. Moreover the quotient of this return transformation over the stable directions should be the map f. We write these transformations as $T^\pm : \Sigma^\pm \to \Sigma_2$ given by

$$(\pm 1, z_2, z_3, V) \mapsto \left(\psi^\pm(z_3), \pm \frac{1}{2} + \frac{z_2}{C}, 2, \Psi^\pm(z_3) V\right)$$

where $C > 0$ is big enough so that the map restricted to the first 3 coordinates is injective and thus a diffeomorphism with its image. We recall that $\psi^\pm$ is part of the definition of the one-dimensional

map f_0. The diffeomorphisms $\Psi^\pm : [0, 1] \to I$ are chosen to ensure that the quotient map is well defined on N, as follows:

$$z_3 \mapsto \Psi^\pm(z_3) := \frac{1}{z_3} \sqrt{\frac{1 - \psi^\pm(z_3)^2}{1 - z_3^{2/\alpha}}}, \quad \text{for } z_3 \in (0, 1)$$

and we set $\Psi^\pm(0) := \mp 1$ and $\Psi^\pm(1) = \pm 1$. We remark that $\Psi^\pm$ are diffeomorphisms, see Fig. 7. Indeed both the numerator and denominator inside the square root can be written as a Taylor series around $z_3 = 1$ with an expression like $const \cdot (1 - z_3) + o(1 - z_3)$ for some non-zero constant, thus $\Psi^\pm$ is differentiable at 1. Now $\psi^\pm$ can be expanded around $z_3 = 0$ as $1 + const \cdot z_3 + o(|z_3|)$ for some positive constant. Hence $\sqrt{1 - (\psi^\pm)^2}$ can be expanded as $(1 - (1 - const \cdot z_3 + o(|z_3|))^2)^{1/2} = const \cdot z_3 + o(|z_3|)$ thus $\Psi^\pm$ is also differentiable at 0.

Now we check that the return map R_0 given by $R_0 := T^\pm \circ L^\pm \circ Y^1 : \Sigma_2 \circlearrowleft$ can be seen as a map on N. Indeed notice that Y^1 commutes with π_1 by construction and that for $x_1 > 0$

$$
\begin{aligned}
\big(T^+ \circ L^+ \circ \pi_1\big)(x_1, x_2, 1, W) &= \big(T^+ \circ L^+\big)\big(x_1, x_2, 1, \big(1 - x_1^2\big)^{1/2} W\big) \\
&= T^+\big(\psi^+\big(|x_1|^\alpha\big), x_2 |x_1|^\beta, |x_1|^\alpha, |x_1|^\alpha \big(1 - x_1^2\big)^{1/2} W\big) \\
&= \bigg(f_0(x_1), \frac{1}{2} + \frac{x_2 |x_1|^\beta}{C}, 2, W \sqrt{1 - \psi^+\big(|x_1|^\alpha\big)^2}\bigg) \\
&= \pi_1\bigg(f_0(x_1), \frac{1}{2} + \frac{x_2 |x_1|^\beta}{C}, 2, W\bigg).
\end{aligned}
$$

Hence we get

$$
\begin{aligned}
R_0(x_1, x_2, 2, W) &= \big(Y^1 \circ T^+ \circ L^+ \circ \pi_1\big)(x_1, x_2, 2, W) \\
&= \pi_1\bigg(f_0(x_1), \frac{1}{2} + \frac{x_2 |x_1|^\beta}{C}, 2, \phi_1(W)\bigg),
\end{aligned}
\tag{7}
$$

where $x_1 > 0$ and $\phi_1 = F$ by the definition of Y^1. Analogous calculations are valid for $x_1 < 0$ taking the maps T^- and L^- into account. By the definition of ϕ_t we get that R_0 maps $\hat{\Sigma} = \Sigma_2 \cap \pi_1(I^2 \times \{1\} \times \mathcal{T})$ inside itself.

Remark 2.5. The contracting direction along the eigendirection of the eigenvalue λ_2 can be made dominated by the other directions in this construction by increasing the contraction rates given by C and λ_2. This is very important to ensure the persistence of the stable lamination, see e.g. [15] and Section 3.1. This is why we assumed the strong dissipative condition $\beta > \alpha + 2$ on the saddle equilibrium s_0. We note that this domination is for the action of the flow we are constructing.

We now remark that quotienting out the contracting directions of $\mathcal{F}^s$ we obtain the map f on N. Indeed, each element of the quotient can be seen as the image of the following projection $\pi_2 : \hat{\Sigma}_2 \to N$ given by $(x_1, x_2, 1, W) \mapsto (x_1, (1 - x_1^2)^{1/2} \pi_{\mathbb{D}}(W))$ where, we recall, $\pi_{\mathbb{D}} : \mathcal{T} \to \mathbb{T}^k = \mathbb{S}^1 \times \overset{k}{\cdots} \times \mathbb{S}^1 \subset \mathbb{C}^k \simeq \mathbb{R}^{2k}$. It is easy to see that *the return map R_0 is semiconjugated to f through π_2*, that is $\pi_2 \circ R_0 = f \circ \pi_2$.

Remark 2.6. We take advantage of the fact that the hyperplanes $\{x_1 = \pm 1\}$ can be identified to a single point due to the dynamics of $\hat{g}_0$. If we do not perform this type of identification, i.e. if we consider instead the projection $\pi_3 : \hat{\Sigma}_2 \to I \times \mathbb{T}^k$ given by $(x_1, x_2, 1, W) \mapsto (x_1, \pi_{\mathbb{D}}(W))$, then we get
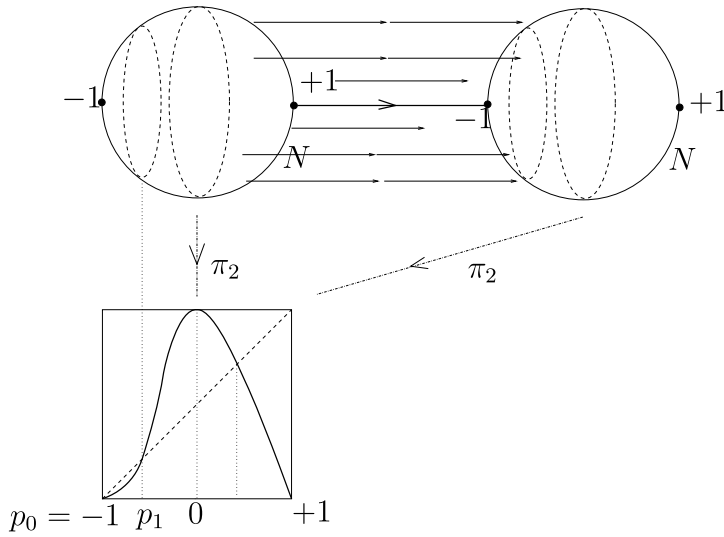
Fig. 8. This represents the initial flow before the perturbation and the one-dimensional quotient.

a cylinder $I \times \mathbb{T}^k$ as the domain of the quotient map, instead of the torusphere N, and likewise $\pi_3 \circ R_0 = (f_0 \times f_1) \circ \pi_3$, where $f_1 : \mathbb{T}^k \circlearrowleft$ is the expanding map on the k-torus defined in Section 2.1.

2.3.1. Localizing some periodic orbits of the flow

The vector field X_0 just constructed has a flow with an attracting periodic orbit, the orbit of $P_0 = \pi_1(-1, y_*, 2, 0)$ for some $y_* \in I$. Indeed, we note that -1 is an attracting fixed point for the one-dimensional map f_0 and that the f_0-orbit of almost every $x \in I$ tends to -1. Then, since on the second coordinate in Σ_2 we have a strong contraction under the action of R_0, this ensures that there exists y_* as above satisfying $R_0(\pi_1(-1, y_*, 2, 0)) = \pi_1(-1, y_*, 2, 0)$. We note that along the x_1 and x_2 directions the flow clearly is a contraction. Moreover, for x in the interval $(-1, p_1)$, the "toruspherical coordinates" of $\pi_1(x, y, 2, W)$ (that is, the last coordinate of dimension $k + 2$) tend to 0 since they are multiplied by $\sqrt{1 - f_0^k(x)^2}$ and $f_0^k(x) \to -1$ as $k \to +\infty$.

In addition the fixed point p_1 of f_0 corresponds to a hyperbolic invariant subset for the flow inside the solid torus $P_1 = \pi_1((p_1, y^*, 2) \times \mathfrak{T})$ for some $y^* \in I$. We observe that quotienting out the stable directions we get the invariant torus $\mathbb{T}_1^k$ as already mentioned before, see Fig. 7.

Finally the orbit of $P_2 = \pi_1(1, \bar{y}, 2, 0)$ is in the stable manifold of P_0 for any $\bar{y} \in I$. Indeed it returns to the stable leaf $\{\pi_1(-1, y, 2, 0): y \in I\}$ and never leaves this leaf in all future returns. In the quotient $\pi_2(\hat{\Sigma}_2)$ the point P_0 is fixed and P_2 is one of its pre-images under f.

2.4. Perturbing the original singular flow

Now we make a perturbation X of the vector field X_0 constructed in the previous section to obtain a Rovella-like attractor.

From now on we assume that P_2 is the point where the component of the unstable manifold of s_0 through Σ^+ first arrives at $I^2 \times \{2\} \times B^{k+2}$. We consider then the positive orbit of P_2 up until it returns to $\hat{\Sigma}_2$. By construction the return point $R_0(P_2)$ has the expression $\hat{P} = \pi_1(-1, \hat{y}, 2, 0)$, so it returns in $\pi_2(-1, 0)$, after the quotient through π_2.

This is a regular orbit, so it admits a tubular neighborhood. We can assume that the orbit chosen above returns to $\hat{\Sigma}_2$ close enough to P_0, that is $\hat{y}$ is close to y_*. We also assume that P_1 is also very close to P_0 so that the tubular neighborhood contains both P_0 and P_1, see Fig. 8.
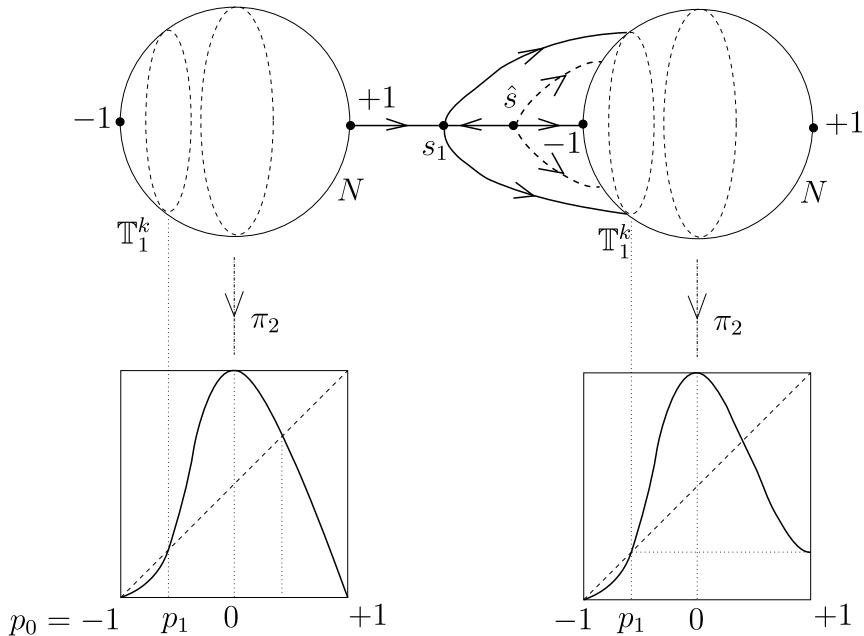
Fig. 9. The perturbed singular flow seen through the projections π_1 and π_2.

We note that since P_0 is a hyperbolic attracting orbit it is easy to extend the tubular neighborhood to its local basin of attraction whose topological closure contains P_1.

In this setting we can now perturb the flow inside the tubular neighborhood in the same way as to produce a kind of Cherry flow, introducing two hyperbolic saddle singularities $\hat{s}$ and s_1. Here the extra dimensions are very useful to enable us to introduce such saddle fixed point with the adequate dimensions of stable and unstable manifolds. The saddle s_1 has $k+1$ expanding eigenvalues $\bar{\lambda}_0, \ldots, \bar{\lambda}_k$, the remaining 4 contracting eigenvalues $\bar{\lambda}_j$, $j = k+1, \ldots, k+4$, and s_1 is sectionally dissipative: $\bar{\lambda}_i + \bar{\lambda}_j < 0$ for all $i \leqslant k$ and $j > k$. In fact the extra dimensions are essential to allow the construction of such a saddle.

The other saddle $\hat{s}$ has $k+2$ expanding eigenvalues and 3 contracting ones. We assume that this perturbation is done in such a way that the $(k+1)$-dimensional unstable manifold of s_1 *contains* $\mathbb{T}_1^k$, see Fig. 9, and the stable manifold of s_1 is everywhere tangent to the subspace given by the direction of the stable manifolds of the solid tori together with the x_2 direction. In this way we can still quotient out the stable leaves, which are preserved by the perturbed flow. *This is the flow of X in the statement of Theorem* A.

Remark 2.7. Since P_2 is part of the unstable manifold of s_0, then we have constructed a saddle-connection between s_0 and s_1. Because the stable manifold of s_1 is 3-dimensional, we can keep the connection for nearby vector fields restricted to a $(k+2)$-codimension sub-manifold of the space of all smooth vector fields: all we have to do is to keep one branch of the one-dimensional unstable manifold of s_0 contained in the 3-dimensional local stable manifold of s_1, and this sub-manifold has codimension $k+2$. See Fig. 10.

The action of the first return map R of the new flow X^t to Σ_2, on the family of stable leaves which project to the interval $[p_1, 1]$, equals (except for a linear change of coordinates) the map g presented in Section 2.1, see Fig. 9. That is, we have a multidimensional non-uniformly expanding transformation as the quotient of the first return map. This transformation sends each k-torus in N to

Fig. 10. The connection between s_0 and s_1.

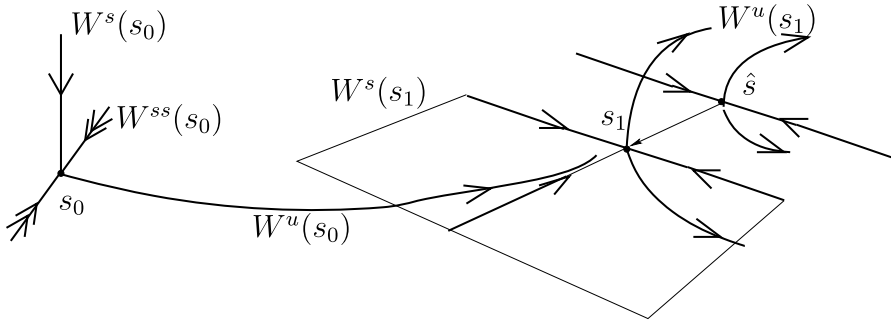another k-torus and so it can be further reduced to the interval map g_0, by considering its action on the tori.

Remark 2.8. The introduction of the saddle-connection changes the return time of the points in Σ_2 under the new flow, but we can assume that the distance from the stable manifold of s_1 of the orbit of $(x_1, x_2, 2, W) \in \Sigma_2$ near s_1 is given by the x_3 coordinate of $L^{\pm}(x_1, x_2, 2, W)$, that is, by $|x_1|^{\alpha}$. Therefore the Poincaré return time is now bounded, after Remark 2.4, by some uniform constant plus $const \cdot \log |x_1|$. The value $|x_1|$ equals the distance of $(x_1, x_2, 2, W)$ to the critical set of g.

This reduction to a one-dimensional model will be essential to our analysis of the existence of an absolutely continuous invariant probability measure for g and the existence of a physical probability measure for the attracting set of the flow near the origin.

2.4.1. The attracting set and an invariant stable foliation

We note that the set $\tilde{\Lambda}_{\Sigma} := \bigcap_{n>0} R^n(\Sigma_2)$ contains a subset $\Lambda_{\Sigma} := \bigcap_{n>0} R^n(\tilde{U})$, where $\tilde{U}$ is a small neighborhood in Σ_2 of $\pi_2^{-1}[p_1, 1]$, which is an attracting set for R, that is, $\overline{R(\tilde{U})} \subset \tilde{U}$. Hence the saturation $\Lambda = \bigcup_{t \in \mathbb{R}} X^t(\Lambda_{\Sigma})$ is also an attracting set for X^t such that $\Lambda_{\Sigma} = \Lambda \cap \Sigma_2$.

It is easy to see that every point $z = \pi_1(x_1, x_2, 1, W)$ of Λ_{Σ} belongs to the solenoid attractor in $\{(x_1, x_2, 1)\} \times B^{k+2}$. Therefore it is straightforward to define a local stable foliation $\mathcal{F}^{ss}$ through the points of Λ_{Σ}: we define $\mathcal{F}^{ss}_z$ to be the stable disk through z of the solid torus $\{(x_1, x_2, 1)\} \times \phi_1(\mathcal{T})$. Hence we get a DR-invariant continuous and uniformly contracting sub-bundle E^{ss} of $T\Lambda_{\Sigma}$ given by $E^{ss}_z = T_z \mathcal{F}^{ss}_z$.

From the existence of the invariant contracting sub-bundle E^{ss} over $T\Lambda_{\Sigma}$, we can define the normal sub-bundle $G = (E^{ss})^{\perp} \cap \mathbb{R} \times \{(0,0)\} \times \mathbb{R}^{k+2}$ to E^{ss} in the tangent space to $I \times \{(x_2, 1)\} \times B^{k+2}$, and use this pair of continuous bundles to define a stable cone field and the complementary cone field

$$C^s(z) = \big\{ (u, v) \in E^{ss}_z \oplus G_z \colon \|u\| \geqslant \|v\| \big\}, \qquad C^u(z) = \big\{ (u, v) \in E^{ss}_z \oplus G_z \colon \|u\| \leqslant \|v\| \big\}$$

for all points $z \in \Lambda_{\Sigma}$.

It is clear that the bundle of tangent spaces to $\{I \times \{(x_2, 1)\} \times B^{k+2}\}_{x_2 \in I}$ is invariant under DR, under both forward and backward iterations. Moreover, we have that C^s is strictly invariant under backward iteration by DR and, consequently, the complementary cone is invariant under forward iteration by DR. Therefore vectors in $C^u(z)$ make an angle with vectors in E^{ss}_z uniformly bounded away from zero. We use this in the next section to prove the existence of a dominated splitting over $T\Lambda_{\Sigma}$.
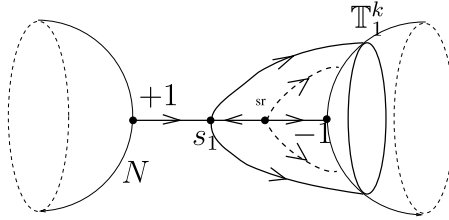
Fig. 11. The last perturbation defining the flow X. We can project the dynamics into a one-dimensional map.

3. Properties of the vector field and its unfolding

We now prove all items of Theorem A and show that the flow of X is chaotic with multidimensional expansion. We also consider its perturbations and prove Theorem B. Some technical points are postponed to Sections 4 and 5 and Appendix A.

3.1. X is chaotic with multidimensional non-uniform expansion

We observe that, since we have a one-dimensional quotient map where the critical point is mapped to a repelling fixed point, we are in the setting of "Misiurewicz maps", see the right-hand side of Fig. 9.

We note that the map on the torusphere for the perturbed flow is *non-uniformly expanding in all directions*, because the non-uniformity is seen on every direction away from the repelling torus $\mathbb{T}_1^k$, and because the singularity contracts also in every direction, see Fig. 11. This was already proved in Section 2.1.

3.1.1. Invariant probability measure

The results in Section 4 ensure that there exists an ergodic absolutely continuous invariant probability measure υ for the transformation g, which is the action of the first return map R of the flow of X to Σ_2 on the stable leaves. Indeed g satisfies all the conditions of Theorem 4.5 since its quotient g_0 is a Misiurewicz map of the interval. Moreover υ is an *expanding measure*: all Lyapunov exponents for υ-a.e. point are strictly positive, as shown in Section 2.1.

From standard arguments using the uniformly contracting foliation through Λ_Σ (recall Section 2.4.1), see e.g. [8, Section 6], we can construct an R-invariant ergodic probability measure ν, whose basin has positive Lebesgue measure on Σ_2, and which projects to υ along stable leaves. Finally, using a suspension flow construction over the transformation R we can easily obtain (see [8, Section 6] for details) a corresponding ergodic physical probability measure μ for the flow of X, which induces ν as the associated R-invariant probability on Σ_2.

We claim that μ is a hyperbolic measure for X^t with $k+1$ positive Lyapunov exponents. To prove this, we first obtain a dominated splitting for the tangent DR of the return map R which identifies the bundle of directions with non-uniform expansion.

3.1.2. Strong domination and smooth stable foliation

We now consider the return map R to Σ_2 and show that the sub-bundle E^s corresponding to the tangent planes to the local stable leaves of the solenoid maps is "strongly γ-dominated" by the non-uniformly expanding direction "parallel" to the toruspheres, for some $\gamma > 1$.

Let $\mathcal{F}_{a,b}^s$ be the bidimensional stable foliation of the solenoid map acting on the section $\{x_1 = a, x_2 = b\} \cap (I^2\{1\} \times B^{k+2})$ for each $(a, b) \in I^2$. Then, from the expression (7) of R_0, which is only modified to R by putting g_0 in the place of f_0, we see that the derivative of R_0 along some leaf $\gamma \in \mathcal{F}_{a,b}^s$ equals the derivative of ϕ_1 along the same direction multiplied by $(1 - g_0(a)^2)^{1/2}$. Since $\|D\phi_1 \mid \gamma\| \leqslant \lambda$ for some constant (not dependent on a, b nor on the particular leaf of $\mathcal{F}_{a,b}^s$) $\lambda \in (0, 1)$, we get that $\|DR \mid \gamma\| \leqslant \lambda(1 - g_0(a)^2)^{1/2}$.

We can estimate expansion/contraction rates of the derivative of the quotient map g as in Section 2.1. Since this map is obtained through a projection of R along $\mathcal{F}^s$, the real expansion and contraction rates of the derivative of R along any direction in the complementary cone C^u are bounded by the corresponding rates of Dg up to constants. These constants depend on the angle between the stable leaves and the direction on the complementary cone field, which is uniformly bounded away from zero.

Hence, recalling that $m(x)$ is the minimum norm of $Dg(x)$ for $x = \pi_1(t, b, 1, z, \Theta)$ with $t \in I$, $z \in \mathbb{D}$ and $\Theta \in [0, 2\pi)^k$, to obtain the smoothness of the foliation, it is enough to get that

$$d(x) := \frac{(1 - g_0(t)^2)^{1/2}}{\|Dg(x)\|^\gamma m(x)^\omega} \tag{8}$$

is bounded by some constant uniformly on every point x, for some $\gamma > 1$ and $\omega > 0$. This implies that for a small enough $\lambda > 0$ we have $\lambda(1 - g_0(t)^2)^{1/2}\|Dg(x)\|^{-\gamma} < \lambda m(x)^\omega < m(x)$, since $\omega > 0$ and $m(x)$ is bounded from above. *This ensures (see* [14, *Theorem* 6.2]*) that $\mathcal{F}^s$ is a C^γ foliation, so that holonomies along the leaves of $\mathcal{F}^s$ are of class C^γ.*

From Section 2.1 we know that

$$m(x) = m(t) = \min\left\{ |g_0'(t)|, 2\left| \frac{\cos(\frac{\pi}{2} g_0(t))}{\cos(\frac{\pi}{2} t)} \right| \right\}, \quad \text{and} \tag{9}$$

$$\|Dg(x)\| = \max\left\{ |g_0'(t)|, 2\left| \frac{\cos(\frac{\pi}{2} g_0(t))}{\cos(\frac{\pi}{2} t)} \right| \right\}. \tag{10}$$

Hence $d(x) = d(t)$ only depends on $t \in I$.

Now we note that for $t \in I \setminus \{-1, 0, 1\}$ the quotient $d(t)$ is continuous. Therefore, if we show that d can be continuously extended to $\{-1, 0, 1\}$, then d is bounded on I and $\lambda d(t)$ can be made arbitrarily small letting the contraction rate λ be small enough, which can be done without affecting the rest of the construction. Having this concludes the proof of the smoothness of $\mathcal{F}^s$.

Finally, we compute, on the one hand $\lim_{t \to \pm 1} |\cos(\frac{\pi}{2} g_0(t)) / \cos(\frac{\pi}{2} t)| = |g_0'(\pm 1)| \neq 0$ which shows that d can be continuously extended to ± 1. On the other hand, by the choice of the map g_0 in Section 2.1, we have that both $m(x)$ and $\|Dg(x)\|$ are of the order of $|t|^{\alpha-1}$ for t near 0 (ignoring multiplicative constants). Thus

$$d(x) = O\left(\frac{(|t|^\alpha)^{1/2}}{(|t|^{\alpha-1})^\gamma (|t|^{\alpha-1})^\omega} \right) = O\left(|t|^{\alpha/2 - (\gamma+\omega)(\alpha-1)} \right). \tag{11}$$

For $1 < \alpha < 2$ we have $\alpha/(2(\alpha-1)) > 1$ so that, in this setting, we can take $\gamma > 1$ and $\omega > 0$ in order that $\alpha/2 - (\gamma + \omega)(\alpha - 1) \geqslant 0$. Hence d can also be extended continuously to 0. This concludes the proof that $\mathcal{F}^s$ is strongly dominated by the action of DR along the directions on the complement C^u of the stable cone field, so that it becomes a C^γ foliation.

Now we define the subspace E to be the sum of the tangent space $E^s := T\mathcal{F}^s$ to $\mathcal{F}^s$ with the x_2 direction E^{ss} on Σ_2, i.e. $E := E^{ss} \oplus E^s$. This bundle E is a DR-invariant contracting sub-bundle which is also strongly dominated by the directions on the center-unstable cone C^u (see Section 2.4.1), since the direction E^{ss} can be made even more strongly contracted than the bundle E^s, see Remark 2.5. In fact, the same argument as above, especially the relation (11), is analogous. The foliation $\mathcal{F}$ tangent to E is then uniformly contracting, with 3-dimensional C^1-leaves and C^γ holonomies, for some $\gamma > 1$.

Remark 3.1. The foliation $\mathcal{F}^{ss}$ is a subfoliation of $\mathcal{F}$ in the sense that every leaf $\gamma \in \mathcal{F}$ admits a foliation $\gamma \cap \mathcal{F}^{ss}$ by leaves of $\mathcal{F}^{ss}$ tangent to E^{ss} at every point.

Remark 3.2. *Arguing with the flow of X, there exists a one-dimensional foliation $\mathcal{F}^{ss}$ tangent to the one-dimensional field of directions $\bigcup_{t \geq 0} DX^t(E_z^{ss})$, since this line bundle is uniformly contracted by X^t. This one-dimensional bundle is also strongly dominated by the "saturated" bundle $\bigcup_{t \geq 0} DX^t(E^s \oplus E^X)$, by the choice of the constants C and β in the construction of the vector field, see Section 2.3.*

We can now complete the construction of the center sub-bundle E^c. The domination just obtained shows that the complementary cone field C^u through the points of Λ_Σ is strictly invariant by forward iteration under DR, so there exists a unique DR-invariant sub-bundle E^c contained in C^u and defined on all points of Λ_Σ. We thus obtain a dominated splitting $E^s \oplus E^c$ of the tangent bundle of Σ_2 over Λ_Σ.

3.1.3. Hyperbolicity of the physical measure

The suspension μ of the ergodic and physical invariant probability measure ν for R is also an ergodic and physical measure for X^t on U. In addition, denoting $\tau(z)$ the Poincaré return time for $z \in \Sigma_2$ (which is a well-defined smooth function except on $\{0\} \times I \times \{1\} \times B^{k+2}$) and $\tau^n(z) = \tau(R^{n-1}(z)) + \cdots + \tau(z)$ for all $n \in \mathbb{Z}^+$ such that $\tau^n(z) < \infty$, we have $DR^n(z) = P_{R^n(z)} \circ DX^{\tau^n(z)} \mid T_z \Sigma_2$, where $P_z : T_z M \to T_z \Sigma_2$ is the projection parallel to the direction $X(z)$ of the flow at $z \in \Sigma_2$. Therefore we can write, for $z \in \Sigma_2$ such that $R^n(z)$ is never in the local stable manifold of s_0 for $n \in \mathbb{Z}^+$ and $v \in E^c(z) \subset T_z \Sigma_2$

$$0 < \limsup_{n \to +\infty} \frac{1}{n} \log \|DR^n(z)v\| \leqslant \limsup_{n \to +\infty} \frac{\tau^n(z)}{n} \cdot \frac{1}{\tau^n(z)} \log \|DX^{\tau^n(z)}(z)v\|. \tag{12}$$

But $\tau^n(z)/n \xrightarrow[n \to \infty]{} \nu(\tau)$ is finite for z in the basin of ν. Indeed, by Remark 2.8, $\tau(z)$ is essentially the logarithm of the distance to the critical set of R, i.e. the intersection of the local stable manifold of s_0 with Σ_2. Indeed, the main contribution to the return time comes from the time it takes $z \in \Sigma_2$ to pass near the singularities, which is given by the logarithm of the distance to $\Sigma_2 \cap W_{loc}^s(s_0)$ and this same value controls the time it takes the point to pass near s_1 also, except for a multiplication by a positive constant, because of the local expression of the flow near a hyperbolic equilibrium and by the form of the connection between s_0 and s_1, see Fig. 10 and Remark 2.8.

In the quotient dynamics of R, i.e. for the map f on N, the function τ is comparable to the logarithm of the distance to the critical set. This function is ν-integrable as a consequence of the non-uniformly expanding properties of the map f, as stated in Theorem 4.5 of Section 4.

Hence (12) implies that the Lyapunov exponents along μ-almost every orbit of X^t are positive along the directions of the bundle $E^{cu}(X^t(z)) := DX^t(E^c(z) \oplus \mathbb{R}X(z))$ for $z \in \Lambda$ with the exception of the flow direction (along which the Lyapunov exponent is zero).

In this way we show that the attractor for the flow of X is chaotic, in the sense that it admits a physical probability measure with $k + 1$ positive Lyapunov exponents.

This concludes the proof of Theorem A.

3.2. Unfolding X

The E^{ss} direction on Σ_2 can be made uniformly contracting with arbitrarily strong contraction rate (see Section 2.3 and Remark 2.5). Moreover, the x_2 direction is dominated by all the other directions under the flow X^t. Thus E^{ss} is a stable direction for the flow over Λ which is dominated by any complementary direction.

Hence this uniformly contracting foliation $\mathcal{F}_X^{ss}$ admits a continuation $\mathcal{F}_Y^{ss}$ for all flows Y^t where Y is close to X in $\mathfrak{X}^2(M)$, that is, in the C^2 topology. Since it is a one-dimensional foliation whose contraction rate can be made arbitrarily small, the holonomies along its leaves are of class C^γ for some $\gamma > 1$, see [14, Theorem 6.2].

This ensures that the return map R_Y to the cross-section $I_\varepsilon \times I_\varepsilon \times \{1\} \times B^{k+2}$, where $I_\varepsilon := (-1 - \varepsilon, 1 + \varepsilon)$, admits a one-dimensional invariant foliation such that the quotient map g_Y of R_Y on the leaves

of this foliation is a $(k+3)$-dimensional C^γ map, for *any vector field Y close to X in the C^2 topology*. In addition, the leaves of $\mathcal{F}^{ss}_Y$ are C^1-close to the leaves of the original $\mathcal{F}^{ss}$ foliation.

Considering the set $I_\varepsilon \times \{(0,1)\} \times B^{k+2}$ diffeomorphic to $I_\varepsilon \times B^{k+2}$, we see that $\mathcal{F}^{ss}_X$ is transverse to this set and thus the continuation remains transverse. Hence we can see the map g_Y as a map between subsets of $I_\varepsilon \times B^{k+2}$. Let $\pi^{ss}_Y : \Sigma_2 \to I_\varepsilon \times B^{k+2}$ be the projection along the leaves of $\mathcal{F}^{ss}_Y$ in what follows and let $\ell := \Sigma_2 \cap W^s_{loc}(s_0(Y))$ be the connected component of the local stable manifold of the continuation of s_0 for Y on the cross-section Σ_2.

We have scarce information about the dynamics of this map: it has a sink $p(Y)$ (the continuation of the sink of R) and is C^γ-close to the quotient of the map R over the foliation $\mathcal{F}^{ss}_X$. Thus, for points outside a neighborhood of $\pi^{ss}_Y(\ell)$ and away from the stable manifold of $p(Y)$, we should have "hyperbolicity" for g_Y due to proximity of R_Y to R, that is, there is a pair of complementary directions on the tangent space such that one is expanded and the other contracted by the derivative of g_Y. The interplay of this hyperbolic-like behavior with the behavior near ℓ is unknown to us.

Conjecture 3. *Similarly to the one-dimensional setting, g_Y admits a physical hyperbolic measure μ_Y for all those vector fields Y which are C^2-close to X and the stable manifold of the sink does not contain the critical region. Moreover the basin of μ_Y should be the complement of the stable manifold of the sink.*

However, we can be more specific along certain sub-manifolds of the space of vector fields, as follows.

3.2.1. Keeping the domination on Σ_2 under perturbation

The argument presented in Section 3.1.2, proving smoothness of a 3-dimensional stable foliation $\mathcal{F}$ after quotienting by π_1, strongly depends on the fact that π_1 identifies every point whose first coordinate is 1, which is represented by an infinite contraction there. We just have to consider (11), which holds because the point 0, corresponding to the intersection of Σ_2 with the local stable manifold of s_0, is sent by g_0 to 1 on each side, that is $g_0(0^\pm) = \lim_{t\to 0^\pm} g_0(t) = 1$.

In order to keep the strong domination for a perturbation Y of X in the C^2 topology, *we restrict the perturbation in such a way that the corresponding points $P_2(Y)$ and $P_3(Y)$ are in the same stable leaf $\xi \in \mathcal{F}_Y$.* This is well defined according to Remark 3.2, see Fig. 7 for the positions of P_2 and P_3. Here $P_2(Y)$ and $P_3(Y)$ are the points of first intersection of each branch $W^u(s_0) \setminus \{s_0\}$ of the one-dimensional unstable manifold of the equilibrium s_0. We note that we can write each branch as an orbit of the flow of Y and so the notion of first intersection with Σ_2 is well defined. This restriction on the vector field corresponds to restricting to a $(k+4)$-codimension sub-manifold $\mathcal{P}$ of the space of vector fields $\mathfrak{X}^2(M)$.

In this way, on the one hand, the same arguments of Section 3.1.2 can be carried through and the strong domination persists for vector fields $Y \in \mathcal{P}$. On the other hand, this implies that there exists an R_Y-invariant 3-dimensional contracting C^γ foliation $\mathcal{F}_Y$ of Σ_2, for some $\gamma > 1$, with C^1 leaves, for all vector fields Y close enough to X within $\mathcal{P}$.

We can then quotient R_Y over the leaves of $\mathcal{F}_Y$ to obtain a $(k+1)$-dimensional map g_Y. We note that, defining the cylinder $\mathcal{C} := I_\varepsilon \times \{0\} \times \{1\} \times e(\mathbb{T}^k \times 0)$ (diffeomorphic to $I_\varepsilon \times \mathbb{T}^k$) inside $I_\varepsilon \times I \times \{1\} \times B^{k+2}$ (recall that $I_\varepsilon = (-1-\varepsilon, 1+\varepsilon)$) we have that the initial foliation $\mathcal{F}$ is everywhere transverse to $\mathcal{C}$. Therefore, since $\mathcal{C}$ is a proper sub-manifold, the continuation $\mathcal{F}_Y$ is still transverse to $\mathcal{C}$ for $Y \in \mathcal{N} \cap \mathcal{P}$. Hence we can define a corresponding quotient map $g_Y : I_\varepsilon \times \mathbb{T}^k \circlearrowleft$ which will *not be*, in general, either a direct or skew-product along the I_ε and $\mathbb{T}^k$ directions.

We observe that g_Y is close to $f_0 \times f_1$ on $I \times \mathbb{T}^k$, recall Remark 2.6 during the construction of the original flow. Hence for pieces of orbits which remain away from a neighborhood of the critical set and away from the basin of the sink, we have uniform expansion in all directions (akin to condition C on the statement of Theorem 4.5).

3.2.2. Keeping the saddle-connection

In addition to keeping the foliation $\mathcal{F}$, we may impose the restriction already mentioned in Remark 2.7 to *keep also the connection between s_0 and s_1*: the component of the unstable manifold of s_0
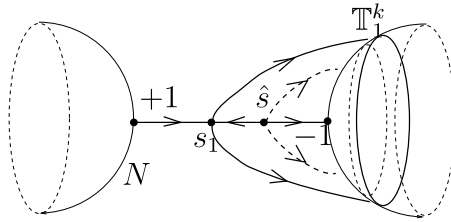
Fig. 12. A symmetrical unfolding. We can still project the dynamics into a one-dimensional map.
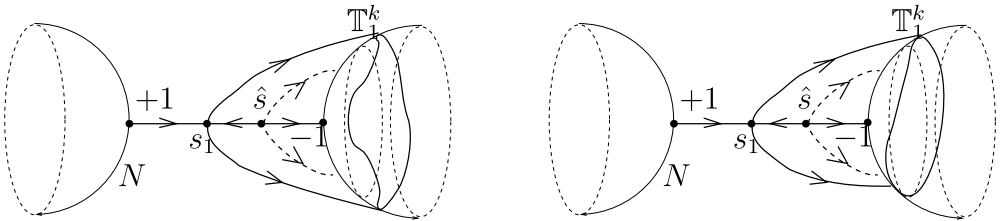

Fig. 13. On the left: a non-symmetrical perturbation. We cannot reduce the analysis to a one-dimensional map. On the right: an open region nearby the singularity is sent into the basin of a sink.

through Σ^+ (i.e. the orbit of $P_2(Y)$) is contained in the stable manifold of s_1, a $(k+1)$-codimension condition on the family of all vector fields Y C^2-close to X. Let $\mathcal{N}$ be the sub-manifold of such vector fields in a neighborhood of X.

Therefore we can ensure that there exists a stable foliation $\mathcal{F}_Y$ nearby $\mathcal{F}_X$ for every vector field $Y \in \mathcal{N} \cap \mathcal{P}$, invariant under the corresponding return map R_Y. We can again quotient R_Y over the leaves of $\mathcal{F}_Y$ to obtain a $(k+1)$-dimensional map g_Y. We observe that $\mathcal{N} \cap \mathcal{P}$ will have codimension $2k + 5$ since the conditions defining $\mathcal{N}$ and $\mathcal{P}$ are independent.

3.2.3. Keeping the one-dimensional quotient map

We can also perturb the vector field X within the manifold $\mathcal{N} \cap \mathcal{P}$ keeping the saddle-connection in such a way that we obtain a one-dimensional C^{1+} quotient map. In this setting we can apply Benedicks–Carleson exclusion of parameters techniques along these families of flows, exactly in the same way Rovella proved his main theorem in [31]. *So we have an analogous result to Rovella's if we perturb the flow keeping the symmetry which allows us to project to a one-dimensional map, that is, if g_Y is a skew-product over I_ε.*

Since we have a well-defined strong stable foliation for the Poincaré map, we can quotient out along such stable foliation obtaining a map in the torusphere. By *keeping the symmetry* we mean that we unfold our Misiurewicz type flow preserving the invariance of each parallel torus in the torusphere. We can consider families which unfold the criticality introduced by the singularity s_1 (as in the right-hand side of Fig. 2) and/or unfold the intersection of the unstable manifold of s_1 with $\mathbb{T}_1^k$, see Fig. 12. Once more, this permits us to reduce the study of the attractor to a one-dimensional problem.

In the very small manifold within $\mathcal{P} \cap \mathcal{N}$ where the one-dimensional quotient map is kept, we can argue just like Rovella in [31] obtaining the same results.

3.2.4. Loosing the one-dimensional quotient map

We note also that, even if we keep the saddle-connection, it is very easy to *perturb this flow* to another arbitrarily close flow such that *the quotient to a one-dimensional map is not defined*, as the left-hand side of Fig. 13 suggests.

3.2.5. Breaking the saddle-connection

Another way to break the one-dimensional quotient is to break the saddle-connection. Then the orbit of $P_2(Y)$ is no longer contained in the stable manifold of s_1. So the positive orbit of $P_2(Y)$ under Y^t follows the unstable manifold of s_1 and crosses Σ_2 at some point, but points in the image $T_Y^+(\Sigma^+)$ near $P_2(Y)$ may no longer be mapped preserving the stable foliation of the solenoids (see Fig. 7). In this case we still have a quotient map $g_Y : I_\varepsilon \times \mathbb{T}^k \circlearrowleft$ but it is no longer a skew-product over I_ε.

3.2.6. Returning away from the sink

We conjecture that, in the situation depicted in the left side of Fig. 13, that is, we have a non-skew-product map g_Y but the image of $P_2(Y)$ under R_Y does not intersect the basin of the sink P_0, *either with or without a saddle-connection* between s_0 and s_1, then it is *always true that there exists an expanding* (*all Lyapunov exponents are positive*) *physical measure with full basin outside the basin of the sink.*

The motivation behind this conjecture is that g_Y is close to g_0 and away from the singular set and away from the basin of the sink we have uniform expansion, since this behavior was present in the original map g_0 and is persistent. Therefore, we have the interplay between expansion and a critical region which will be approximately a circle on the cylinder $I_\varepsilon \times \mathbb{T}^k$. This setting is similar to the one introduced by Viana in [37, Theorem A] and carefully studied in e.g. [6,2,5].

3.2.7. Returning inside the basin of the sink

On the one hand, the perturbation can send points nearby P_2 into the basin of the sink P_0, *either with or without a saddle-connection* between s_0 and s_1, as shown in the right-hand side of Fig. 13. We prove, in Section 5, that this implies that the *basin of the sink grows to fill the whole manifold* Lebesgue modulo zero, for such vector fields in the sub-manifold $\mathcal{P} \cap \mathcal{N}$.

This completes the proof of Theorem B except for the last item proved in Section 5.

4. Higher-dimensional Benedicks–Carleson conditions

To present the statement and proof of the existence of an absolutely continuous invariant probability measure on the setting of higher-dimensional maps, we need to recall some notions from non-uniformly expanding dynamics.

4.1. Non-flat critical or singular sets

Let f be a C^{1+} local diffeomorphism outside a compact proper sub-manifold $\mathcal{S}$ of M with positive codimension. The set $\mathcal{S}$ may be taken as some set of critical points of f or a set where f fails to be differentiable. The sub-manifold $\mathcal{S}$ has an at most countable collection of connected components $\{\mathcal{S}_i\}_{i \in \mathbb{N}}$, which may have different codimensions. This is enough to ensure that the volume or Lebesgue measure of $\mathcal{S}$ is zero and, in particular, that f is a *regular map*, that is, if $Z \subset M$ has zero volume, then $f^{-1}(Z)$ has zero volume also.

In what follows we assume that the number of connected components of $\mathcal{S}$ is finite. It should be possible to drop this condition if we impose some global restrictions on the behavior of the map f, see [24] for examples with one-dimensional ambient manifold M of non-uniformly expanding maps with infinitely many critical points. We do not pursue this issue in this work.

We say that $\mathcal{S} \subset M$ is a *non-flat critical or singular set* for f if the following conditions hold for each connected component $\mathcal{S}_i$. The first one essentially says that f *behaves like a power of the distance* to $\mathcal{S}_i$: there are constants $B > 1$ and real numbers $\alpha_i > \beta_i$ such that $\alpha_i - \beta_i < 1, 1 + \beta_i > 0$ and *on a neighborhood U_i of $\mathcal{S}_i$* (where $U_i \cap U_j = \emptyset$ if $j \neq i$ and we also take $U_i \subset B(\mathcal{S}_i, 1/2)$) for every $x \in U_i$

(S1)
$$\frac{1}{B} \operatorname{dist}(x, \mathcal{S})^{\alpha_i} \leqslant \frac{\|Df(x)v\|}{\|v\|} \leqslant B \operatorname{dist}(x, \mathcal{S})^{\beta_i} \quad \text{for all } v \in T_x M.$$

Moreover, we assume that the functions $\log|\det Df(x)|$ and $\log\|Df(x)^{-1}\|$ are *locally Lipschitz* at points $x \in U_i$ with Lipschitz constant depending on $\text{dist}(x, \mathcal{S})$: for every $x, y \in U_i$ with $\text{dist}(x, y) < \text{dist}(x, \mathcal{S}_i)/2$ we have

(S2) $$\left|\log\|Df(x)^{-1}\| - \log\|Df(y)^{-1}\|\right| \leqslant \frac{B}{\text{dist}(x, \mathcal{S}_i)^{\alpha_i}} \, \text{dist}(x, y);$$

(S3) $$\left|\log|\det Df(x)| - \log|\det Df(y)|\right| \leqslant \frac{B}{\text{dist}(x, \mathcal{S}_i)^{\alpha_i}} \, \text{dist}(x, y).$$

The assumption that the number of connected components is finite implies that there exists $\beta > 0$ such that $\max_i\{|\alpha_i|, |\beta_i|\} \leqslant \beta$ and we also assume that for all $x \in M \setminus \mathcal{S}$

(S4) $$\frac{1}{B} \, \text{dist}(x, \mathcal{S})^{\beta} \leqslant \frac{\|Df(x)v\|}{\|v\|} \leqslant B \, \text{dist}(x, \mathcal{S})^{-\beta} \quad \text{for all } v \in T_x M.$$

The case where $\mathcal{S}$ is equal to the empty set may also be considered. The assumption $1 + \beta_i > 0$ prevents that the image of arbitrary small neighborhoods around the singular set accumulates every point of M (consider e.g. the Gauss map $[0, 1] \ni x \mapsto x^{-1} \bmod 1$) which would prevent a meaningful definition of "singular value", see the statement of Theorem 4.5 in what follows.

4.2. Hyperbolic times

We write $x_i = f^i(x)$ for $x \in M$, $i \geqslant 0$, and also $S_n^f \varphi(x) = S_n \varphi(x) = \sum_{i=0}^{n-1} \varphi(x_i)$, $n \geqslant 1$, for the ergodic sums of a function $\varphi : M \to \mathbb{R}$ with respect to the action of f, in what follows. For the next definition it will be useful to introduce $\text{dist}_\delta(x, \mathcal{S})$, the δ-*truncated* distance from x to $\mathcal{S}$, defined as $\text{dist}_\delta(x, \mathcal{S}) = \text{dist}(x, \mathcal{S})$ if $\text{dist}(x, \mathcal{S}) \leqslant \delta$, and $\text{dist}_\delta(x, \mathcal{S}) = 1$ otherwise. From now on we write $\psi(x) = \log\|Df(x)^{-1}\|$ and $\mathfrak{D}_r(x) = -\log\text{dist}_r(x, \mathcal{S})$ for $x \in M \setminus \mathcal{S}$ and $r > 0$.

Let $\beta > 0$ be given by the non-flat conditions on $\mathcal{S}$, and fix $b > 0$ such that $b < \min\{1/2, 1/(4\beta)\}$. Given $c > 0$ and $\delta > 0$, we say that h is an (e^c, δ)-*hyperbolic time* for a point $x \in M$ if, for all $0 \leqslant k \leqslant h$,

$$S_k \psi(x_{n-k}) \leqslant -ck \quad \text{and} \quad S_k \mathfrak{D}_\delta(x_{n-k}) \leqslant bck. \tag{13}$$

We convention that an empty sum evaluates to 0 so that the above inequalities make sense for all indexes in the given range.

We say that the *frequency of (e^c, δ)-hyperbolic times* for $x \in M$ is greater than $\theta > 0$ if, for infinitely many times n, there are $h_1 < h_2 < \cdots < h_\ell \leqslant n$ which are (e^c, δ)–hyperbolic times for x and $\ell \geqslant \theta n$.

The following statement summarizes the main properties of hyperbolic times. For a proof the reader can consult [4, Lemma 5.2] and [4, Corollary 5.3].

Proposition 4.1. *There are $0 < \delta_1 < \delta/4$ and $C_1 > 0$ (depending only on δ and σ) such that if h is an (e^c, δ)-hyperbolic time for x, then there is a* hyperbolic neighborhood V_x of $x = x_0$ in M for which

(1) f^h *maps V_x diffeomorphically onto the ball of radius δ_1 around x_n;*
(2) *for $1 \leqslant k \leqslant h$ and $y, z \in V_x$, $\text{dist}(y_{n-k}, z_{n-k}) \leqslant C_1 e^{ck/2} \text{dist}(y_h, z_h)$;*
(3) *for all $y \in V_x$, $S_h \mathfrak{D}_\delta(y) \leqslant S_h \mathfrak{D}_\delta(x) + o(\delta)$ where $o(\delta)/\delta \to 0$ when $\delta \to 0$;*
(4) $f^h \mid V_x$ *has distortion bounded by C_1: if $y, z \in V_x$, then $\frac{|\det Df^h(y)|}{|\det Df^h(z)|} \leqslant C_1$.*

Remark 4.2. The image of V_x by f^h is away from $B(\mathcal{S}, 3\delta/4)$: for $y \in V_x$

$$\text{dist}(y_h, \mathcal{S}) \geqslant \text{dist}(x_h, \mathcal{S}) - \text{dist}(y_h, x_h) \geqslant \delta - \delta_1 > \frac{3}{4}\delta$$

since $\text{dist}_\delta(x_h, \mathcal{S}) \geqslant 1$.

Item (3) above, which is not found in [4], is an easy consequence of item (2): for every $1 \leqslant k \leqslant h$ and $y \in V_x$

$$-\log \frac{\mathrm{dist}_\delta(y_{h-k}, \mathcal{S})}{\mathrm{dist}_\delta(x_{h-k}, \mathcal{S})} \leqslant -\log \frac{\mathrm{dist}_\delta(x_{h-k}, \mathcal{S}) - \mathrm{dist}(y_{h-k}, x_{h-k})}{\mathrm{dist}_\delta(x_{h-k}, \mathcal{S})}$$

$$= -\log\left(1 - \frac{\mathrm{dist}(y_{h-k}, x_{h-k})}{\mathrm{dist}_\delta(x_{h-k}, \mathcal{S})}\right)$$

$$\leqslant -\log\left(1 - \frac{\delta_1 \sigma^{k/2}}{e^{bck}}\right) \leqslant -\log\left(1 - \frac{3}{4}\delta e^{(1/2-b)ck}\right)$$

$$\leqslant \frac{(3/4)\delta}{1 - (3/4)\delta} \cdot \frac{3}{4}\delta e^{(1/2-b)ck}.$$

So each point y in a hyperbolic neighborhood as above satisfies

$$S_h \mathfrak{D}_\delta(y) \leqslant S_h \mathfrak{D}_\delta(x) + \frac{(3/4)^2 \delta^2}{1 - (3/4)\delta} \sum_{j=0}^{h} e^{(1/2-b)cj} \leqslant S_h \mathfrak{D}_\delta(x) + \frac{(3/4)^2 \delta^2}{1 - (3/4)\delta} \cdot \frac{1}{1 - e^{bc}},$$

and the last term is $o(\delta)$.

In what follows we say that an open set V is a *hyperbolic neighborhood* of any one of its points $x \in V$, with $(e^{c/2}, \delta)$-*distortion time* h (or, for short, an $(e^{c/2}, \delta)$-*hyperbolic neighborhood*), if the properties stated in items (1) through (4) of Proposition 4.1 are true for x and $k = h$.

We also say that a given point x has *positive frequency of hyperbolic neighborhoods* bounded by $\theta > 0$ if there exist $c, \delta > 0$ and neighborhoods V_{h_k} of x with (e^c, δ)-distortion time h_k, for all $k \geqslant 1$, such that for every big enough k we have $k \geqslant \theta h_k$.

4.3. Existence of many hyperbolic neighborhoods versus absolutely continuous invariant probability measures

Hyperbolic times appear naturally when f is assumed to be *non-uniformly expanding* in some set $H \subset M$: there is some $c > 0$ such that for every $x \in H$ one has

$$\liminf_{n \to +\infty} \frac{1}{n} S_n \psi(x) < -c, \tag{14}$$

and points in H satisfy some *slow recurrence to the critical or singular set*: given any $\varepsilon > 0$ there exists $\delta > 0$ such that for every $x \in H$

$$\limsup_{n \to +\infty} \frac{1}{n} S_n \mathfrak{D}_\delta(x) \leqslant \varepsilon. \tag{15}$$

The next result has been proved in [4, Theorem C and Lemma 5.4]. It provides sufficient conditions for the existence of many hyperbolic times along the orbit of points satisfying the non-uniformly expanding and slow recurrence conditions.

Theorem 4.3. *Let $f : M \to M$ be a C^{1+} local diffeomorphism outside a non-flat critical or singular set $\mathcal{S} \subset M$. If there is some set $H \subset M \setminus \mathcal{S}$ such that (14) and (15) hold for all $x \in H$, then for any given $0 < \xi < 1$ and $0 < \zeta < bc$ there exist $\delta > 0$ and $\theta > 0$ such that the frequency of $(e^{-c\xi}, \delta)$-hyperbolic times for each point $x \in H$ is bigger than θ. Moreover for such hyperbolic times h we have $S_h \mathfrak{D}_\delta(x) \leqslant \zeta h$.*

Together with Proposition 4.1 the results from Theorem 4.3 ensure the existence of positive frequency of hyperbolic neighborhoods around Lebesgue almost every point.

This will imply the existence of absolutely continuous invariant probability measures for the map f through the following result from [27].

Theorem 4.4. *Let $f : M \to M$ be a C^{1+} local diffeomorphism outside a non-degenerate exceptional set $\mathcal{S} \subset M$. If there are $c, \delta > 0$ such that the frequency of (e^c, δ)-hyperbolic neighborhoods is bigger than $\theta > 0$ for Lebesgue almost every $x \in M$, then f has some absolutely continuous invariant probability measure.*

Since we are using a modified definition of hyperbolic time, we present a proof of Theorems 4.3 and 4.4 in Appendix A for completeness.

4.4. Existence of absolutely continuous probability measures

The following theorem provides higher-dimensional existence result for physical measures which applies to our setting.

Theorem 4.5. *Let $f : M \setminus \mathcal{S} \to M$ be a C^{1+} local diffeomorphism, where $\mathcal{S}$ is a compact sub-manifold of M which is a non-flat critical/singular set for f. We define*

$$f(\mathcal{S}) := \bigcap_{n \geqslant 1} \overline{f\big(B_{1/n}(\mathcal{S})\big)},$$

the set of all accumulation points of sequences $f(x_n)$ for x_n converging to $\mathcal{S}$ as $n \to +\infty$, and assume that f also satisfies:

- A: *f is non-uniformly expanding along the orbits of critical values:* there exist $c_0 > 0$ and $N \geqslant 1$ such that for all $x \in f(\mathcal{S})$ and $n \geqslant N$ we have $S_n \psi(x) \leqslant -c_0 n$;
- B: *the critical set has slow recurrence to itself:* given $\varepsilon > 0$ we can find $\delta > 0$ such that for all $x \in f(\mathcal{S})$ there exists $N = N(x)$ satisfying $S_n \mathfrak{D}_\delta(x) \leqslant \varepsilon n$ for every $n \geqslant N$;
- C: *f is uniformly expanding away from the critical/singular set:* for every neighborhood U of $\mathcal{S}$ there exist $c = c(U) > 0$ and $K = K(U) > 0$ such that for any $x \in M$ and $n \geqslant 1$ satisfying $x = x_0, x_1, \ldots, x_{n-1} \in M \setminus U$, then $S_n \psi(x) \leqslant K - cn$;
- D: *f does not contract too much when returning near the critical/singular set:* that is, there exist $\kappa > 0$ and a neighborhood $\hat{U}$ of $\mathcal{S}$ such that for every open neighborhood $U \subset \hat{U}$ of $\mathcal{S}$ and for $x = x_0$ satisfying $x_1, \ldots, x_{n-1} \in M \setminus U$ and either $x_0 \in U$ or $x_n \in U$, then $S_n \psi(x_0) \leqslant \kappa$.

Then f has an absolutely continuous invariant probability measure ν such that $\mathfrak{D}_d$ is ν-integrable for some (and thus all) $d > 0$.

We observe that condition B above ensures, in particular, that $f(\mathcal{S}) \cap \mathcal{S} = \emptyset$, for otherwise $\mathfrak{D}_\delta(x)$ is not defined for $x \in \mathcal{S} \cap f(\mathcal{S})$. Moreover, condition C above is just a convenient translation to this higher-dimensional setting of the conclusion of the one-dimensional theorem of Mañé [17], ensuring uniform expansion away from the critical set and basins of periodic attractors. It can be read alternatively as: given $\delta > 0$ there are $C, \lambda > 0$ such that if $x_i \in M \setminus B(\mathcal{S}, \delta)$ for $i = 0, \ldots, n - 1$, then $\|Df^n(x)^{-1}\| \leqslant Ce^{-\lambda n}$. In addition, condition D is a translation to our setting of a similar property that holds for unidimensional multimodal "Misiurewicz maps", that is, for maps whose critical orbits are non-recurrent, ensuring a minimal lower bound for the derivative of the map along orbits which return near $\mathcal{S}$.

Now we show that under the conditions in the statement of Theorem 4.5, we can find a full measure subset of points of M having positive density of hyperbolic times.

Theorem 4.6. *Let $f : M \setminus \mathcal{S} \to M$ be a C^{1+} local diffeomorphism away from a non-flat critical/singular set $\mathcal{S}$, satisfying all conditions in the statement of Theorem 4.5. Then for every small enough $0 < \xi < 1$ there exist $\delta = \delta(\xi) > 0$ and $\theta = \theta(\xi, \delta) > 0$ such that Lebesgue almost every $x \in M$ admits positive frequency bounded by θ of $(e^{-\xi c_0}, \delta)$-hyperbolic neighborhoods.*

From this result we deduce Theorem 4.5 applying Theorem 4.4. So all we need to do is prove Theorem 4.6. For the integrability of $\mathfrak{D}$ see Remark 4.14 in what follows.

4.4.1. Existence of hyperbolic neighborhoods

Fix $\xi_0, \varepsilon, \tilde{\delta} > 0$ and small enough so that condition B is satisfied in what follows and $\xi_0 c_0 < b$. Let $\zeta > 0$ be small enough in order that

$$\frac{\xi_0 c_0}{1 + \alpha_i} + 2\zeta < \frac{\xi_0 c_0}{1 + \zeta} \quad \text{for all } i. \tag{16}$$

Depending on f and ξ_0, we can choose the pair $(\varepsilon, \tilde{\delta})$ so that, from conditions A and B above together with Theorem 4.3, every point $z \in f(\mathcal{S})$ has infinitely many $(e^{-2\xi_0 c_0}, \tilde{\delta})$-hyperbolic times $h_1 < h_2 < h_3 < \cdots$ satisfying

$$S_{h_i} \mathfrak{D}_{\tilde{\delta}}(z) \leqslant \zeta h_i \quad \text{for } i \geqslant 1. \tag{17}$$

Then there are corresponding hyperbolic neighborhoods V_i of z satisfying the conclusions of Proposition 4.1 for each hyperbolic time h_i of z, where $\tilde{\delta} > 0$ does not depend on $z \in \mathcal{S}$, that is $f^{h_i} \mid V_i : V_i \to B(z_{h_i}, \delta_1)$ is a diffeomorphism with a ball of radius $\delta_1 \in (0, \tilde{\delta}/4)$ whose inverse is a contraction with rate bounded by $e^{-\xi_0 c_0 h_i}$.

We will consider, instead of V_i, the subset $B_i \subset V_i$ given by

$$B_i = \left(f^{n_i} \mid V_i \right)^{-1} \left(B(z_n, \delta_2) \right) \tag{18}$$

where we set $2\delta_2 = \delta_1$. Since $f(\mathcal{S})$ is compact, we can cover this set by finitely many hyperbolic neighborhoods of the type B_i.

Now we fix a connected component $\mathcal{S}_i$ of $\mathcal{S}$.

4.4.2. Hyperbolic neighborhoods near the critical/singular set

Let T_i be the smallest distortion (or hyperbolic) time associated to the balls covering $f(\mathcal{S}_i)$. We remark that T_i can be taken arbitrarily big, independently of ξ_0, ζ, $\tilde{\delta}$, because every $z \in f(\mathcal{S}_i)$ has positive frequency of hyperbolic neighborhoods and, consequently, the open neighborhoods in the above covering can be made arbitrarily small.

We observe that, by definition of hyperbolic neighborhoods, if $\mathrm{dist}(z, f(\mathcal{S}_i)) < e^{-bj\xi_0 c_0}$ for some $j \geqslant 1$ and there exists $x \in f(\mathcal{S}_i)$ such that $z \in V_i(x)$, then the corresponding distortion time h_i satisfies $h_i \geqslant j$.

We note also that, by the non-flat condition (S1) on f near $\mathcal{S}_i$, a ρ-neighborhood of $\mathcal{S}$ is sent into a $\rho^{1+\beta_i}$-neighborhood of $f(\mathcal{S}_i)$, for each $\rho > 0$. Indeed, since $\mathcal{S}_i$ is assumed to be a sub-manifold, we can find for x, on a tubular neighborhood of $\mathcal{S}_i$ with radius ρ, a curve $\gamma : [0, \rho] \to M$ from $\gamma(0) \in \mathcal{S}_i$ to $\gamma(1) = x$ such that $\mathrm{dist}(\gamma(t), \mathcal{S}_i) = t$ and $\|\dot{\gamma}(t)\| = 1$ for $t \in [0, \rho]$. Hence

$$\mathrm{dist}\left(f(x), f(\mathcal{S}_i) \right) \leqslant \int_0^\rho \left\| Df\left(\gamma(t) \right) \dot{\gamma}(t) \right\| dt \leqslant \int_0^\rho B \, \mathrm{dist}\left(\gamma(t), \mathcal{S} \right)^{\beta_i} dt$$

$$= \frac{B \rho^{1+\beta_i}}{1 + \beta_i} = \frac{B}{1 + \beta_i} \, \mathrm{dist}(x, \mathcal{S})^{1+\beta_i}. \tag{19}$$

Therefore we can find $C_2 = C_2(\delta_2) > 0$ such that $C_2^{1+\beta_i} B/(1+\beta_i) = \delta_2$ and if, for some $j \geqslant 0$

$$\text{dist}(x, \mathcal{S}_i) \leqslant C_2 e^{-\xi_0 c_0 (T_i + j)/(1+\alpha_i)} := d_j, \tag{20}$$

then the smallest possible distortion time h of $f(x)$ is at least $\frac{1+\beta_i}{1+\alpha_i}(T_i + j)$, since

$$\delta_2 e^{-\xi_0 c_0 h} \leqslant \text{dist}\big(f(x), f(\mathcal{S}_i)\big) \leqslant \delta_2 e^{-\xi_0 c_0 \frac{1+\beta_i}{1+\alpha_i}(T_i + j)} \leqslant \delta_2 e^{-b\xi_0 c_0 \frac{1+\beta_i}{1+\alpha_i}(T_i + j)}.$$

Note that by the previous observations and Remark 4.2 we have

$$f\big(B(\mathcal{S}_i, \tilde{\delta})\big) \subset B\left(f(\mathcal{S}_i), \frac{B}{1+\beta_i}\tilde{\delta}^{1+\beta_i}\right)$$

and we can assume without loss of generality that

$$d_0 < \tilde{\delta} - \delta_1, \tag{21}$$

letting T_i grow if necessary.

In the opposite direction, we can find an upper bound for the distortion time associated with a given distance to $\mathcal{S}_i$ reversing the inequality in (19) as follows. Letting γ denote a smooth curve $\gamma : [0, 1] \to M$ such that $\gamma(0) \in \mathcal{S}_i$ and $\gamma(1) = x$ for any given fixed x near $\mathcal{S}_i$, we get

$$\text{dist}\big(f(x), f(\mathcal{S}_i)\big) = \inf_{\gamma} \int_0^1 \big\| Df\big(\gamma(t)\big)\dot{\gamma}(t)\big\| \, dt \geqslant \frac{1}{B} \inf_{\gamma} \int_0^1 \text{dist}\big(\gamma(t), \mathcal{S}_i\big)^{\alpha_i} \big\|\dot{\gamma}(t)\big\| \, dt$$

$$\geqslant \frac{1}{B} \inf_{\gamma} \int_0^1 \text{dist}\big(\gamma(t), \mathcal{S}_i\big)^{\alpha_i} \left|\frac{d}{dt}\text{dist}\big(\gamma(t), \mathcal{S}\big)\right| \, dt$$

$$= \frac{1}{B} \inf_{\gamma} \int_0^1 \left|\frac{d}{dt}\frac{\text{dist}(\gamma(t), \mathcal{S}_i)^{1+\alpha_i}}{1+\alpha_i}\right| \, dt$$

$$= \frac{1}{B(1+\alpha_i)} \inf_{\gamma} \text{var}_{t \in [0,1]} \text{dist}\big(\gamma(t), \mathcal{S}\big)^{1+\alpha_i} \geqslant \frac{\text{dist}(x, \mathcal{S})^{1+\alpha_i}}{B(1+\alpha_i)}, \tag{22}$$

where we use, beside the non-flat condition (S1), the relation

$$\left|\frac{d}{dt}\text{dist}\big(\gamma(t), \mathcal{S}_i\big)\right| = \big\|\pi_t\big(\dot{\gamma}(t)\big)\big\| = \big\|\dot{\gamma}(t)\big\| \cdot \big|\cos \angle\big(\dot{\gamma}(t), N_t\big)\big| \leqslant \big\|\dot{\gamma}(t)\big\|$$

and write N_t for the normal direction to the level sub-manifold

$$S_t = \big\{z \in M \colon \text{dist}(z, \mathcal{S}_i) = \text{dist}\big(\gamma(t), \mathcal{S}_i\big)\big\} \quad \text{at } \gamma(t);$$

and π_t for the orthogonal projection from $T_{\gamma(t)}M$ to N_t. We also use the well-known relation $\text{var}_{[0,1]} \varphi = \int_0^1 |D\varphi(t)| \, dt$ for the total variation of a differentiable function $\varphi : [0, 1] \to \mathbb{R}$.

From the inequality (22) we see that if $d_j \geqslant \text{dist}(x, \mathcal{S}) \geqslant d_{j+1}$ and $f(x) \in B_i$ for some hyperbolic neighborhood with distortion time h, then

$$\delta_2 e^{-\xi_0 c_0 h} \geqslant \text{dist}\big(f(x), f(\mathcal{S}_i)\big) \geqslant \frac{C_2^{1+\alpha_i}}{B(1+\alpha_i)} e^{-\xi_0 c_0 (T_i+j+1)} = \frac{C_2^{1+\beta_i+\alpha_i-\beta_i}}{B(1+\alpha_i)} e^{-\xi_0 c_0 (T_i+j+1)}$$

$$= \delta_2^{\alpha_i-\beta_i} \frac{(1+\beta_i)^{\alpha_i-\beta_i}}{B^{1+\alpha_i-\beta_i}(1+\alpha_i)} e^{-\xi_0 c_0 (T_i+j+1)} \geqslant C_3 \delta_2^{\alpha_i-\beta_i} e^{-\xi_0 c_0 (T_i+j+1)} \quad \text{or}$$

$$h \leqslant (T_i+j+1) - \frac{\log C_3}{\xi_0 c_0} + \frac{1-(\alpha_i-\beta_i)}{\xi_0 c_0} \log \delta_2 \leqslant \left(1+\frac{\zeta}{2}\right)(T_i+j+1)$$

as long as T_i is big enough, depending on f and ξ_0, c_0, δ_2. We remark that we have used the condition $0 < \alpha_i - \beta_i < 1$ in the inequalities above. For future reference we write these inequalities (for a big enough T_i) in the convenient format

$$d_j \geqslant \text{dist}(x, \mathcal{S}_i) \geqslant d_{j+1}, \quad j \geqslant 0 \quad \Longrightarrow \quad \frac{1+\beta_i}{\xi_0 c_0} \mathfrak{D}_{d_0}(x) \leqslant h \leqslant \frac{(1+\zeta/2)(1+\alpha_i)}{\xi_0 c_0} \mathfrak{D}_{d_0}(x). \quad (23)$$

We are now ready for the main arguments.

Claim 4.7. *There are $(e^{\xi c_0}, \delta)$-hyperbolic neighborhoods for each point in $B(\mathcal{S}_i, \delta) \setminus \mathcal{S}_i$, for suitable constants $\xi, \delta \in (0, 1)$.*

Indeed, for each $y_0 \in B(\mathcal{S}_i, d_0) \setminus \mathcal{S}_i$ there exists a unique integer k such that $y_0 \in B(\mathcal{S}_i, d_k) \setminus B(\mathcal{S}_i, d_{k+1})$. By the choice of k we know that $y_1 = f(y_0)$ has some distortion time $\frac{1+\beta_i}{1+\alpha_i}(T_i+k) \leqslant h \leqslant (1+\zeta/2)(T_i+k+1)$. Using the non-flat conditions on $\mathcal{S}_i$ we can estimate the norm of the derivative and the volume distortion in a suitable neighborhood of y_0, and show that y_0 has a hyperbolic neighborhood with $h+1$ as distortion time, with slightly weaker constants of expansion and distortion, as follows.

Let B_i be the hyperbolic neighborhood containing y_1 with distortion time h (as defined in (18)). The image of B_i under f^h is the δ_2-ball around z_h for some point $z_0 \in f(\mathcal{S}_i)$ and y_{h+1} is inside this ball. Note that since $\text{dist}(y_1, f(\mathcal{S}_i)) \geqslant d_{k+1}^{1+\alpha_i}/(B(1+\alpha_i))$ from (22), then

$$\text{dist}(y_{h+1}, z_h) \geqslant \frac{d_{k+1}^{1+\alpha_i}}{B(1+\alpha_i)} e^{\xi_0 c_0 h} = \frac{C_2^{1+\alpha_i}}{B(1+\alpha_i)} e^{\xi_0 c_0 h \left(1-\frac{T+k+1}{h}\right)} \geqslant C_3 \delta_2^{\alpha_i-\beta_i}. \quad (24)$$

Thus if we set $2\delta_3 = C_3 \delta_2^{\alpha_i-\beta_i}$, then we can take a hyperbolic neighborhood of y_1 defined by $W := (f^h \mid V_i)^{-1}(B(y_{n+1}, \delta_3))$, where V_i is the original neighborhood associated to B_i, see (18). Observe that because diam $W \leqslant \delta_3 e^{-\xi_0 c_0 h}$

$$\text{dist}\big(W, f(\mathcal{S}_i)\big) \geqslant \frac{C_2^{1+\alpha_i}}{B(1+\alpha_i)} d_{k+1}^{1+\alpha_i} - \delta_3 e^{-\xi_0 c_0 h}$$

$$= \big(C_3 \delta_2^{\alpha_i-\beta_i} d_{k+1}^{1+\alpha_i} e^{\xi_0 c_0 h} - \delta_3\big) e^{-\xi_0 c_0 h} \geqslant \delta_3 e^{-\xi_0 c_0 h}$$

by the definition of δ_3.

Now we find a radius $\rho > 0$ such that the image $f(B(y_0, \rho))$ covers the hyperbolic neighborhood W of y_1. By the definition of k and the non-flatness condition (S1) we have that $f(B(y_0, \rho)) \supset B(y_1, \rho_1)$ for all small enough $\rho > 0$, where $\rho_1 \geqslant \rho \|Df(y_0)^{-1}\|^{-1} \geqslant B^{-1} \rho d_{k+1}^{-\alpha_i}$. Since we want $\rho_1 \geqslant \delta_3 e^{-\xi_0 c_0 h}$ it is enough that

$$\frac{\rho}{B} e^{-\frac{\alpha_i}{1+\alpha_i} \xi_0 c_0 (T+k+1)} \geqslant \delta_3 e^{-\xi_0 c_0 h} \quad \Longleftrightarrow \quad \rho \geqslant B \delta_3 e^{-\xi_0 c_0 \frac{T+k+1}{1+\alpha_i} \left((1+\alpha_i)\frac{h}{T+k+1} - \alpha_i\right)}.$$

But using the relations obtained between h and $T + k + 1$ we get

$$\rho \geqslant B\delta_3 e^{-\xi_0 c_0 \frac{T+k+1}{1+\alpha_i}((1+\alpha_i)(1+\zeta/2)-\alpha_i)} = B\delta_3 d_{k+1}^{1+(1+\alpha_i)\zeta/2}.$$

We note that $\gamma := (1 + \alpha_i)\zeta/2$ is positive due to the non-flatness conditions. If we take $w, \tilde{w} \in B(y_0, d_{k+1}^{1+\gamma})$, then

$$\text{dist}(w, S_i) \geqslant \text{dist}(y_0, S_i) - \text{dist}(w, y_0) \geqslant d_{k+1} - d_{k+1}^{1+\gamma} = d_{k+1}(1 - d_{k+1}^{\gamma}) > d_{k+2} \qquad (25)$$

whenever T_i is big enough in order that $1 - d_{k+1}^{\gamma} > d_{k+2}/d_{k+1} = e^{-\xi_0 c_0/(1+\alpha_i)}$. Thus if $w_1 = f(w)$ and $\tilde{w}_1 = f(\tilde{w})$ are both in W, then using the bound for the inverse of the derivative provided also by the non-flatness condition (S1)

$$\begin{aligned}
\text{dist}(w, \tilde{w}) &\leqslant Bd_{k+2}^{-\alpha_i} \text{dist}(w_1, \tilde{w}_1) \leqslant Bd_{k+2}^{-\alpha_i} e^{-\xi_0 c_0 h} \text{dist}(w_{h+1}, \tilde{w}_{h+1}) \\
&= B\exp\left(-\xi_0 c_0 (h+1)\left(\frac{h}{h+1} - \frac{\alpha_i}{1+\alpha_i}\frac{T_i + k + 2}{h+1}\right)\right) \text{dist}(w_{h+1}, \tilde{w}_{h+1}) \\
&\leqslant e^{-\xi_0 c_0(1 - \frac{\alpha_i}{1+\beta_i})(h+1)} \text{dist}(w_{h+1}, \tilde{w}_{h+1}) \qquad (26)
\end{aligned}$$

for all big enough h since $h + 1 > \frac{1+\beta_i}{1+\alpha_i}(T_i + k) + 1 > \frac{1+\beta_i}{1+\alpha_i}(1 + o(T_i))(T_i + k + 2)$ where $o(T_i)$ is a small quantity which tends to zero when T_i is taken arbitrarily large. We recall that condition (S1) on α_i, β_i ensures that $1 + \beta_i > \alpha_i$, so that $0 < \xi_1 := \xi_0(1 - \frac{\alpha_i}{1+\beta_i})$ and the contraction rate is bounded by $e^{-\xi_1 c_0}$. In particular this shows that we can indeed take a neighborhood with the radius ρ we estimated above.

This means that every point w_1 in W is the image of some point $w \in B(y_0, d_{k+1}^{1+\gamma})$, and the connected component W_0 containing y_0 of the pre-image of W under f is fully contained in $M \setminus B(S_i, d_{k+2})$, by the relation (25). Thus the neighborhood W_0 of y_0 satisfies items (1) and (2) of Proposition 4.1 for $n = h + 1$ iterates and for an $e^{-\xi_1 c_0}$ contraction rate. Therefore item (3) of the same proposition also holds as a consequence with $\delta = d_0$, see Section 4.2.

In addition we can estimate

$$\begin{aligned}
S_{h+1}\mathfrak{D}_{d_0}(y_0) = \mathfrak{D}_{d_0}(y_0) + S_h\mathfrak{D}_{d_0}(y_1) &\leqslant -\log d_{k+1} + \zeta h + o(\tilde{\delta}) \\
&\leqslant \left(\xi_0 c_0 \frac{T_i + k + 1}{(h+1)(1+\alpha_i)} + \zeta\frac{h}{h+1} + \frac{o(\tilde{\delta})}{h+1}\right)(h+1) \\
&\leqslant \left(\frac{\xi_0 c_0}{1+\alpha_i} + 2\zeta\right)(h+1) \leqslant \frac{\xi_0 c_0}{1+\zeta}(h+1), \qquad (27)
\end{aligned}$$

where we have used item (3) of Proposition 4.1 applied to the hyperbolic neighborhood W of y_1 and the choice of ζ in (16) and (17). This bound is essential to obtain positive frequency of hyperbolic neighborhoods in the final stage of the proof. We use here the assumption B of the statement of Theorem 4.5 with appropriately chosen constants.

Moreover we also have the following estimate for the bounded distortion of volume, again using the non-flat conditions together with the above inequalities for $w, \tilde{w} \in W_0$

$$\log \frac{|\det Df(w)|}{|\det Df(\tilde{w})|} \leqslant B\,\text{dist}(W_0, S_i)^{-\alpha_i} \text{dist}(w, \tilde{w}) \leqslant e^{-\xi_1 c_0(h+1)} \text{dist}(w_{h+1}, \tilde{w}_{h+1}).$$

Therefore we can bound (using that $f(w), f(\tilde{w}) \in B_i$ and Proposition 4.1)

$$\log \frac{|\det Df^{h+1}(w)|}{|\det Df^{h+1}(\tilde{w})|} \leqslant \sum_{j=0}^{h} \log \frac{|\det Df(w_j)|}{|\det Df(\tilde{w}_j)|}$$

$$\leqslant \sum_{j=0}^{h} e^{-\xi_1 c_0 j} \operatorname{dist}(w_{h+1}, \tilde{w}_{h+1}) \leqslant \frac{\delta_3}{1 - e^{-\xi_1 c_0}} = C_1.$$

Hence W_0 satisfies all the conditions of Proposition 4.1 with appropriate constants.

We stress that the value of T_i (thus the value of d_0) depends only on α_i, β_i and ξ_0, c_0, δ_1.

This completes the proof of Claim 4.7.

Remark 4.8. The minimal backward contraction in $B(S_i, d_0)$ near a singularity can be made arbitrarily big in a single iterate taking T_i large enough or, which is the same, taking the neighborhood of the singularities small enough.

Remark 4.9. The constant of average expansion ξc_0 is the same in all cases for each connected component of S, but in principle the radius $\delta_3 = \delta_3(i)$ and the distance $d_0 = d_0(i)$ to S_i ensuring existence of distortion times depend on the connected component. However we assume that the values of T_i are big enough so that all the inequalities above are satisfied for all connected components S_i and such that the corresponding neighborhoods of S_i be contained in the $\tilde{\delta}$-neighborhood of S.

In what follows we write δ (which is smaller than $\tilde{\delta}$) for the smallest value $d_0(i)$ over all connected components.

4.4.3. Hyperbolic neighborhoods for almost every point

We use the statement of Claim 4.7 from now on and show that all points whose orbit does not fall into the singular/critical set admit some distortion time with well-defined contraction rate and slow approximation rate.

Claim 4.10. *Lebesgue almost every point admits an $(e^{-\xi_2 c_0}, \delta)$-hyperbolic neighborhood for some iterate ℓ and the frequency of visits of the ℓ-iterates to a δ-neighborhood of S is bounded by $\frac{\xi_0 c_0}{1+\varsigma}$.*

In a similar way to the one-dimensional multimodal case, we use condition D on $\hat{U}$ to show that we can obtain a minimal derivative when an orbit returns to any fixed arbitrarily small neighborhood $U \subset \hat{U}$ of the critical/singular set. From this intermediate result we deduce that most points on $M \setminus U$ have some distortion time. We prove first an auxiliary result.

Claim 4.11. *There exists a minimal average expansion rate $e^{-c/2}$ either for the first return of x_0 to a small enough neighborhood U of S, or for the first N iterates, where N does not depend on the starting point $x_0 \in U$.*

Let $U = B(S, \delta)$ be an open neighborhood of S compatible with the choices of the T_i as in the proofs of the previous claims. So U is the union of a number of connected open sets, one for each connected component of S.

The first h iterates of x_0 correspond to an $(e^{-\xi c_0}, \delta)$-distortion time. From condition C there are $K, c > 0$ such that for $n \geqslant h$ satisfying $x_h, \ldots, x_{n-1} \in M \setminus U$ we have that $S_n \psi(x_0) \leqslant -\xi c_0 h + K - c(n - h)$.

We shrink the neighborhood U so that the smaller distortion time $\underline{h}$ for points $x_0 \in U \setminus S$ satisfies

$$\frac{\kappa}{\underline{h}} < \min\left\{\frac{\xi c_0}{4}, \frac{c}{8}\right\} \quad \text{and} \quad \frac{\underline{h}}{\underline{h} + 2K/c} \geqslant \frac{1}{2}. \tag{28}$$

We write $[t] := \max\{i \in \mathbb{Z} \colon i \leqslant t\}$ for all $t \in \mathbb{R}$ in what follows and set $N = [2K/c]$.

Case (i) – *the orbit returns to U in more than N iterates:* In this case we have for $n > N$ that $S_n \psi(x_0) \leqslant -\frac{c}{2} n$, since we can assume without loss that $c < \xi c_0$. Notice that we have the same conclusion if the orbit of x_0 *never returns* to U.

Case (ii) – *the return to U occurs in less than N iterates:* We now use condition D to get, since the first return iterate n satisfies $n \geqslant h \geqslant \underline{h}$

$$S_n \psi(x_0) \leqslant -\xi c_0 h + \kappa = n\left(-\xi c_0 \frac{h}{n} + \frac{\kappa}{n}\right) \leqslant n\left(-\frac{\xi c_0}{2} + \frac{\xi c_0}{4}\right) = -\frac{\xi c_0}{4} n,$$

since $\underline{h}$ was chosen as in (28).

This completes the proof of Claim 4.11, setting $\min\{c/2, \xi c_0/4\}$ as the average expansion rate.

Now we prove Claim 4.10.

Again fix an arbitrary $x_0 \in M \setminus U$ such that the orbit of x_0 never falls into $\mathcal{S}$. We divide the argument in two cases.

Case (iii) – *the orbit takes more than N iterates to enter U:* Arguing as in the proof of Case (i) above we get for all $n > N$ satisfying $x_0, \ldots, x_{n-1} \in M \setminus U$ that $S_n(x_0) \leqslant K - cn \leqslant -\frac{c}{2} n$. This implies that there exists some distortion time $h \leqslant n$ for x_0 by the proof of Theorem 4.3. We note that in this case $S_n \mathfrak{D}_\delta(x_0) = 0$.

Case (iv) – *the orbit enters U in at most N iterates:* Let $j \leqslant N$ be the first entrance time of the orbit of x_0 in U, h the distortion time associated to x_j and use Claim 4.11 to get

$$S_{j+h} \psi(x_0) \leqslant \kappa + S_h \psi(x_j) \leqslant \kappa - \frac{c}{2} h = (j+h)\left(\frac{\kappa}{j+h} - \frac{c}{2}\frac{h}{j+h}\right)$$

$$\leqslant (j+h)\left(\frac{c}{8} - \frac{c}{4}\right) = -\frac{c}{8}(j+h)$$

since $h \geqslant \underline{h}$ by the choice of U in (28).

We also obtain that $S_{j+h} \mathfrak{D}_\delta(x_0) = S_h \mathfrak{D}_\delta(x_j) \leqslant \frac{\xi_0 c_0}{1+\zeta} h \leqslant \frac{\xi_0 c_0}{1+\zeta}(j+h)$.

Moreover the distortion of f^{j+h} on the connected component of $(f^j)^{-1}(V_{x_j})$ containing x_0 is bounded from above by a constant dependent on N only (since we are dealing with a local diffeomorphism away from a given fixed neighborhood of $\mathcal{S}$), where V_{x_j} is the hyperbolic neighborhood of x_j given by Claim 4.7.

This completes the proof of Claim 4.10 if we set ξ_2 such that $\xi_2 c_0 = c/8$.

4.4.4. Positive frequency of hyperbolic neighborhoods almost everywhere

Here we finish the proof of Theorem 4.6.

Claim 4.12. *The frequency of $(e^{-\xi_2 c_0}, \delta)$-hyperbolic neighborhoods is positive and bounded away from zero Lebesgue almost everywhere.*

We now define an auxiliary induced map $F : \tilde{M} \to M$, so that the F-iterates correspond to iterates of f at distortion times, as follows. For $x_0 \in \tilde{M}$ we have two cases:

Expansion without shadowing: the $\ell \geqslant N$ iterates of x belong to $M \setminus U$. In this case there exists some $(e^{-\xi_2 c_0}, \delta)$-distortion time $\ell \leqslant N$ and we define $F(x) = f^\ell(x)$ and $\tau(x) = \ell = q(x)$.

Expansion with shadowing: let $0 \leqslant q < N$ be the least non-negative integer such that $x_q \in U$ and p be the $(e^{-\xi_0 c_0}, \delta)$-distortion time associated to x_q from Claim 4.7. We define $F(x) = f^{p+q}(x) = x_{p+q}$ and $\tau(x) = p + q$ and $q(x) = q$ in this case.

The images of F always belong to $M \setminus B(\mathcal{S}, \delta)$ by the choice of δ from (21) following Remark 4.2. In addition, for any given $x_0 \in \tilde{M}$ the map $F(x)$ is defined and the iterate $\tau(x)$ has all the properties of an $(e^{-\xi_2 c_0}, \delta)$-distortion time for x_0 according to Claim 4.10.

Now we observe the statement of Claim 4.12 is a consequence of the following property: there exists $\theta > 0$ such that for every $x \in \tilde{M}$

$$\limsup_{n \to +\infty} \frac{1}{n} \#\{0 \leqslant j < n : f^j(x) \in \mathcal{O}_F^+(x)\} \geqslant \theta, \tag{29}$$

where $\mathcal{O}_F^+(x) = \{F^i(x), \ i \geqslant 0\}$ is the positive orbit of the induced map F. Moreover it is easy to see that for each $x \in M$ and each $n \in \mathbb{N}$

$$\#\{0 \leqslant j < n : f^j(x) \in \mathcal{O}_F^+(x)\} = \sup\left\{k \geqslant 0 : S_{k+1}^F \tau(x) = \sum_{i=0}^{k} \tau(F^i(x)) < n\right\}. \tag{30}$$

We remark that from (30) to obtain (29) it is enough to show that $S_{k+1}^F \tau(x) < \frac{1}{\theta} k$, at least for every big enough k. Indeed $\{k \geqslant 0 : S_{k+1}^F \tau(x) < n\} \supset \{k \geqslant 0 : \frac{1}{\theta} k < n\}$ so $\sup\{k \geqslant 0 : S_{k+1}^F \tau(x) < n\} \geqslant \theta n$.

The bounds (23) and (27) together with the definition of F ensure that we are in the conditions of the following result.

Lemma 4.13. *Assume that we have an induced map $F = f^\tau$ for some $\tau : K \to \mathbb{N}$ defined on a positive invariant subset K such that for every $x \in K$:*

(1) *$\tau(x) = q(x) + p(f^{q(x)}(x))$ for well-defined integer functions q on K and p on $f^{q(x)}(x)$ for all $x \in K$;*
(2) *there exists $N \in \mathbb{N}$ such that $q \leqslant N$;*
(3) *there exist $0 < d < 1$ and $C, \rho > 0$ such that $0 < \rho C < 1$ and for all $x \in K$*
    (a) *the iterates $x, f(x), \ldots, f^{q(x)-1}(x)$ are outside $B_d(\mathcal{S})$;*
    (b) *$p(f^{q(x)}(x)) \leqslant C\mathfrak{D}_d(f^{q(x)}(x))$;*
    (c) *$S_p^f \mathfrak{D}_d(f^{q(x)}(x)) \leqslant \rho p$.*

Then $S_k^F \tau(x) \leqslant \frac{N}{1-\rho C} k$ for each $x \in K$ and every $k \geqslant 1$.

In addition, we observe that since f is a regular map (that is $f_* \mathrm{Leb} \ll \mathrm{Leb}$) then we can further assume that the full Lebesgue measure set $\tilde{M}$ is f-invariant, since $\bigcap_{i \geqslant 0} f^{-i}(\tilde{M})$ also has full Lebesgue measure. Hence we can apply Lemma 4.13 with $K = \tilde{M}$, $C = \frac{1+\zeta/2}{\xi_0 c_0}$ and $\rho = \frac{\xi_0 c_0}{1+\zeta}$ to obtain $\theta \geqslant \frac{\zeta}{2(1+\zeta)\ell}$.

Remark 4.14. According to item (3)(b) of the statement of Lemma 4.13 above, if we have an absolutely continuous invariant probability measure ν for f, then $\nu(\mathfrak{D}_d) \leqslant \rho < \infty$ and so $\mathfrak{D}_d$ is ν-integrable, as claimed in Theorem 4.5.

This completes the proof of Theorem 4.6 except for the proof of Lemma 4.13.

Proof of Lemma 4.13. Using the definition of F and the assumptions on τ, for every given $k \geqslant 0$ and $x \in K$ we can associate a sequence $q_0, p_0, q_1, p_1, \ldots, q_k, p_k$ such that for each $i = 0, \ldots, k$ we have $q_i = q(F^i(x))$ and $q_i + p_i = \tau(F^i(x))$. This together with the assumptions of item (3) in the statement of the lemma allows us to estimate

$$S_k^F \tau(x) = \sum_{i=0}^{k} (q_i + p_i) \leqslant \sum_{i=0}^{k-1} \left(N + C\mathfrak{D}_{d_0}\left(f^{q_i}(F^i(x))\right)\right)$$

$$\leqslant kN + C \sum_{i=0}^{k-1} S_{q_i+p_i}^f \mathfrak{D}_{d_0}\big(F^i(x)\big) \tag{31}$$

$$\leqslant kN + C\rho \sum_{i=0}^{k-1} (q_i + p_i) = kN + C\rho S_k^F \tau(x), \tag{32}$$

where in (31) we have used that $\mathfrak{D}_{d_0} \geqslant 0$ and that this function equals zero at each of the q_i iterates before each visit to $B(\mathcal{S}, d)$. The contraction in (32) implies $S_k^F \tau(x) \leqslant \frac{N}{1-\rho C} k$ as stated. $\quad\Box$

5. Periodic attractor with full basin of attraction

Here we prove item (3) of Theorem B. We show that a perturbation of X, like the one depicted in the right side of Fig. 13, for a flow $Y \in \mathcal{P} \cap \mathcal{N}$ is such that U is as a trapping region which coincides Lebesgue modulo zero with the basin of a periodic attracting orbit (a sink) of Y. This is a consequence of the smoothness of the first return map to Σ_1 after quotienting out the stable leaves, which is the reason why we assume the flow is of class at least C^2 and restrict the vector field to a sub-manifold $\mathcal{P} \cap \mathcal{N}$ of all possible vector fields nearby X, together with the robustness of property C in the statement of Theorem 4.5.

Indeed, let $g = g_Y$ be the action on the stable leaves of the first return map R_Y of the flow of $Y \in \mathcal{P} \cap \mathcal{N}$ to Σ_2. Recall that we constructed X having on $\Lambda_\Sigma = \bigcap_{n \in \mathbb{Z}^+} g^n(\Sigma_2)$ a partially hyperbolic splitting so that the stable foliation does persist under perturbations. The projection along the leaves of this foliation in Σ_2 is absolutely continuous with Hölder Jacobian, as a consequence of the strong domination obtained in Section 3.1.2.

The map g sends a δ-neighborhood of $\mathcal{S}$ into the local basin of attraction of a periodic sink p. We denote by B the stable set of the orbit of p. On $N \setminus B$ the map g is uniformly expanding, since $B \supset B(\mathcal{S}, \delta)$ and condition C on the statement of Theorem 4.5 is persistent under small perturbations.

Lemma 5.1. *Lebesgue almost every point of N belongs to the basin of the periodic sink.*

Proof. Arguing by contradiction, assume that the basin B of the sink is such that $E := N \setminus B$ has positive volume. Since B contains a neighborhood of σ_1 and a neighborhood of the sink, then E is an invariant subset satisfying condition C of the statement of Theorem 4.5. Hence $f_1 \mid E$ is uniformly expanding: there exist $N \geqslant 1$ and $\lambda \in (0, 1)$ such that $\|(Dg_Y^N)^{-1}\| \leqslant \lambda$. Therefore, since g_Y is log-Hölder and expanding, and E is closed, invariant and has positive volume, we can apply the arguments in [3] to show that there exists a ball U of radius $r > 0$ fully contained in E.

We claim that for $g = g_Y^N$ there exists $\rho > 1$ such that $g^k(U)$ contains a ball of radius $\rho^k r$ for all $k \geqslant 1$, which yields a contradiction, since the ambient manifold is compact and E is by assumption a proper subset.

To prove the claim, recall that g is a local diffeomorphism on a neighborhood of E, since E is far from the singularities of the stable foliation. We assume that $B(x_0, s_0)$ is a ball centered at x_0 with radius s_0 contained in E and consider $g(B(x_0, s_0))$.

Let us take y_1 in the boundary of $g(B(x_0, s_0))$ and a smooth curve $\gamma_1 : [0, 1] \to N$ such that $\gamma_1(0) = x_1 := g(x_0)$ and $\gamma_1(1) = y_1$. Let γ_0 be a lift of γ_1 under g, that is, $\gamma_1 = g \circ \gamma_0$ such that $\gamma_0(0) = x_0$. We then define $s := \sup\{t \in [0, 1]: \ \gamma_1([0, t]) \subset g(B(x_0, s_0))\}$. Clearly $s > 0$ and by its definition and the expansion properties of g in E we get

$$\lambda \times \big(\text{length of } \gamma_1\big([0, s]\big)\big) \geqslant \text{length of } \gamma_0\big([0, s]\big) \geqslant \text{dist}\big(\gamma_0(0), \gamma_0(s)\big).$$

However, $\gamma_1(s)$ is at the boundary of $g(B(x_0, s_0))$ so that $\gamma_0(s)$ is also at the boundary of $B(x_0, s_0)$, because g is a local diffeomorphism. Thus we get

$$\text{dist}(y_1, x_1) \geqslant \text{length of } \gamma_1([0, s]) \geqslant \frac{1}{\lambda} \times \text{dist}(\gamma_0(0), \gamma_0(s)) \geqslant \frac{1}{\lambda} s_0$$

and the claim is proved with $\rho = \lambda^{-1}$. $\quad\square$

Acknowledgments

Appendix A. Non-uniform expansion and existence of hyperbolic times

Here we prove Theorem 4.3. The proof is very similar to [4, Lemma 5.4] but our definition of hyperbolic times/hyperbolic neighborhoods is slightly different, in a crucial way, from the definition on [4], and we include a proof for completeness.

We start with the following extremely useful technical result will be the key for several arguments.

Lemma A.1. *Let $H \geqslant c_2 > c_1 > 0$ and $\zeta = (c_2 - c_1)/(H - c_1)$. Given real numbers $a_1, \ldots, a_N$ satisfying*

$$\sum_{j=1}^{N} a_j \geqslant c_2 N \quad \text{and} \quad a_j \leqslant H \quad \text{for all } 1 \leqslant j \leqslant N,$$

there are $l > \zeta N$ and $1 < n_1 < \cdots < n_l \leqslant N$ such that

$$\sum_{j=n+1}^{n_i} a_j \geqslant c_1 \cdot (n_i - n) \quad \text{for each } 0 \leqslant n < n_i, \ i = 1, \ldots, l.$$

Proof. See [18, Lemma 11.3]. $\quad\square$

Proof of Theorem 4.3. The proof uses Lemma A.1 twice, first for the sequence $a_j = -\psi(x_{j-1})$ (properly cut off so that it becomes bounded from above), and then for $a_j = \mathfrak{D}_\delta(x_j)$ for an adequate $\delta > 0$.

Let $H \subset M \setminus \mathcal{S}$ be such that conditions (14) and (15) hold for all $x \in H$ and let $0 < \xi < 1$ and $\zeta > 0$ be given, and take $x = x_0 \in H$ and $\gamma_0 := (2 + \xi)/3 \in (\xi, 1)$, $\gamma_2 := (1 - \xi)/3$ and $\gamma_3 = \gamma_1 - \gamma_2 = (1 + 2\xi)/3$. Then for every large N we have $S_N \psi(x) \leqslant -\gamma_0 cN$. Moreover since $f(\mathcal{S}) \cap \mathcal{S} = \emptyset$ we can assume that $\zeta < \inf\{\text{dist}(x, y): x \in \mathcal{S}, \ y \in f(\mathcal{S})\}$.

For any fixed $\rho > \beta$, by non-degeneracy condition (S1), we can find a neighborhood V of $\mathcal{S}$ such that

$$|\psi(z)| \leqslant \rho \mathfrak{D}(z) \quad \text{for every } x \in V. \tag{33}$$

Setting $\varepsilon_1 > 0$ such that $\rho \varepsilon_1 \leqslant \gamma_1$, we can use the slow approximation condition to find $r_1 > 0$ so small that

$$S_N \mathfrak{D}_{r_1}(x) \leqslant \varepsilon_1 N. \tag{34}$$

We may assume without loss that $V = B(\mathcal{S}, r_1)$ in what follows. Now we fix $H_1 \geqslant \max\{c, \rho|\log r_1|, \sup_{M \setminus V} |\psi|\}$ and define the set $E = \{z \in M: \psi(z) < -H_1\}$ and the sequence $a_j = -(\psi \chi_{M \setminus E})(x_{j-1})$.

We remark that there is a shift between the index of a_j and that of x_{j-1} in the above definition.

We note that by construction $a_j \leqslant H_1$ and that $x_j \in E$ implies $x_j \in V$, $\mathfrak{D}(x_j) < -\log r_1$, because $\rho|\log r_1| \leqslant H_1 < -\psi(x_j) < \rho \mathfrak{D}(x_j)$.

This means that $\mathfrak{D}_{r_1}(x_j) = \mathfrak{D}(x_j) < |\log r_1|$ whenever $x_j \in E$. From (33) and (34) we get that $-S_N(\psi \chi_E)(x) \leqslant \rho S_N(\mathfrak{D} \chi_E)(x) \leqslant \rho \varepsilon_1 N \leqslant \gamma_1 c N$.

Therefore $\sum_{j=1}^N a_j = -S_N \psi(x) - S_N(\psi \chi_E)(x) \geqslant (\gamma_0 - \gamma_1) c N = \gamma_3 c N$. Hence we can apply Lemma A.1 with $c_2 = \gamma_3 c$, $c_1 = \xi c$, $H = H_1$, obtaining $\theta_1 = \gamma_2 c / (H - \xi c) \in (0, 1)$ and $l_1 \geqslant \theta_1 N$ times $1 \leqslant p_1 < \cdots < p_{l_1} \leqslant N$ such that

$$\sum_{j=n+1}^{p_i} \psi(x_{j-1}) \leqslant - \sum_{j=n+1}^{p_i} a_j \leqslant -\xi c (p_i - n) \tag{35}$$

for every $0 \leqslant n < p_i$ and $1 \leqslant i \leqslant l_1$.

Let now $\varepsilon_2 > 0$ be small enough so that $\varepsilon_2 < \min\{\zeta, bc\theta_1\}$, and let $r_2 > 0$ be such that $-S_N \mathfrak{D}_{r_2}(x_1) \geqslant -\varepsilon_2 N$ from the slow recurrence condition (we note that $\text{dist}(x_0, \mathcal{S}) > \zeta > r_2$). Taking $c_1 = bc$, $c_2 = -\varepsilon_2$, $A = 0$, and $\theta_2 = \frac{c_2 - c_1}{A - c_1} = 1 - \frac{\varepsilon_2}{bc}$ we can apply again Lemma A.1 to $a_j = -\mathfrak{D}_{r_2}(x_j)$.

We remark that now there is no shift between the index of a_j and x_j in the previous definition.

In this way we obtain $l_2 \geqslant \theta_2 N$ times $1 \leqslant q_1 < \cdots < q_{l_2} \leqslant N$ such that

$$\sum_{j=n+1}^{q_i} \mathfrak{D}_{r_2}(x_j) \leqslant \varepsilon_2 (q_i - n) \leqslant \zeta (q_i - n) \tag{36}$$

for every $0 \leqslant n < q_i$ and $1 \leqslant i \leqslant l_2$.

Finally since our choice of θ_2 ensures that $\theta = \theta_1 + \theta_2 - 1 > 0$, then there must be $l = (l_1 + l_2 - N) \geqslant \theta N$ and $1 \leqslant n_1 < \cdots < n_l \leqslant N$ such that (35) and (36) simultaneously hold.

This exactly means that we have condition (13) with $\delta = r_2$, because $\mathfrak{D}_{r_2}(x_0) = 0$ by the choice of r_2 above, and ξc as the logarithm of the contraction rate, as in the statement of Theorem 4.3. □

Appendix B. Solenoid by isotopy

We recall that $\mathbb{S}^1 = \{z \in \mathbb{C}: |z| = 1\}$, $\mathbb{T} = (\mathbb{S}^1)^k$, $B^k := \{x = (x_1, \ldots, x_k) \in \mathbb{R}^n: \sum_{i=1}^k x_i^2 = 1\}$ for all $k \geqslant 1$, and $\mathcal{T}^k = \mathbb{T}^k \times \mathbb{D}$, where $\mathbb{D} = \{z \in \mathbb{C}: |z| < 1\}$.
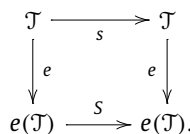
Here we prove the results needed in Section 2.3 ensuring the existence of a smooth family of embeddings of $\mathcal{T}^k$ into B^{k+2}, for all $k \geqslant 1$, which deforms a tubular neighborhood in B^{k+2} of the usual embedding of $\mathbb{T}^k$ into $B^{k+1} \simeq B^{k+1} \times \{0\} \subset B^{k+2}$, into the embedding of $\mathcal{T}$ into the image of the Smale solenoid map.

More precisely, consider the identity map i on $\mathcal{T}$ and the solenoid map

$$s : \mathcal{T} \to \mathcal{T}, \quad \big(\Theta = (\theta_1, \ldots, \theta_k), z\big) \mapsto \big(\Theta^2 = (\theta_1^2, \ldots, \theta_k^2), A_\Theta(z)\big)$$

where A_Θ is a contraction with contraction rate bounded by $0 < \lambda < 1$, and the map in the coordinate Θ is the expanding torus endomorphism f_1 defined in Section 2.1, but restricted to $\mathbb{T}^k \subset \mathbb{C}^k$.

Proposition B.1. *There exists an embedding $e : \mathcal{T} \to B^{k+2}$ such that the projections $\pi_{\mathbb{D}} : \mathcal{T} \to \mathbb{T}^k$ on the first coordinate and $\tilde{\pi}_{\mathbb{D}} : e(\mathcal{T}) \to e(\mathbb{T}^k \times \{0\})$ along the leaves of the foliation $\mathcal{F}^s := \{e(\Theta \times \mathbb{D})\}_{\Theta \in \mathbb{T}^k}$ of $e(\mathcal{T})$ define the solenoid map S by the commutative diagram:*

$$
\begin{array}{ccc}
\mathcal{T} & \xrightarrow{\ \ s\ \ } & \mathcal{T} \\
{\scriptstyle e}\downarrow & & \downarrow{\scriptstyle e} \\
e(\mathcal{T}) & \xrightarrow{\ \ S\ \ } & e(\mathcal{T}).
\end{array}
$$

Moreover there exists a smooth family $e_t : \mathcal{T}^k \to B^{k+2}$ of embeddings for all $t \in [0, 1]$ such that $e_0 = e \circ i$ and $e_1 = e \circ s$.

We prove this statement in the following steps.

B.1. An embedding of $\mathbb{T}^k$ into B^{k+2}

We argue by induction on $k \geqslant 1$. We know how to embed $\mathbb{T}^1$ in B^3. Let us denote by e^1 this embedding and fix a small number $d > 0$ and $\lambda \in (0, 1/2)$.

We assume that we have an embedding e^l of $\mathbb{T}^l$ on B^{l+2} for all $l = 1, \ldots, k - 1$, in such a way that the image of e^{l+1} is in a tubular neighborhood of the image of e^l inside B^{l+2} with size $\leqslant d\lambda^{l+1}$, for $1 \leqslant l < k - 1$.

For each $w \in e^{k-1}(\mathbb{T}^{k-1}) \subset \mathbb{R}^{k+1} \simeq \mathbb{R}^{k+1} \times 0 \subset \mathbb{R}^{k+2}$ let $N_w = (T_w \mathbb{S}_k)^{\perp}$ be the normal space to $e^{k-1}(\mathbb{T}^{k-1})$ at w in $\mathbb{R}^{k+2}$ (where we take in $\mathbb{R}^{k+2}$ the usual Euclidean inner product). This is a 3-dimensional space.

We know we can embed $\mathbb{T}^1$ into B^3 through e^1. So by a simple rescaling we can assume that e^1_w embeds $\mathbb{T}^1$ into a small neighborhood of w in $w + N_w$. To keep the inclusion in the tubular neighborhood of the image of e^{k-1}, we take this neighborhood around w to have radius $\lambda^{k+1}d$.

Hence letting $\hat{w} \in \mathbb{T}^{k-1}$ be the unique element such that $e^{k-1}(\hat{w}) = w$ and considering the map $e^k : \mathbb{T}^{k-1} \times \mathbb{T}^1 \to B^{k+2}$ given by $(\hat{w}, \theta) \mapsto e^1_w(\theta)$ we easily see that

- $D_1 e^k(\hat{w}, \theta)(\mathbb{R}^{k-1}) = De^{k-1}(\hat{w})(\mathbb{R}^{k-1})$ is the tangent space of $e^{k-1}(\mathbb{T}^{k-1})$ at w;
- $D_2 e^k(\hat{w}, \theta)(\mathbb{R})$ is a subspace of N_w.

Therefore the tangent map De^k to e^k always has maximal rank and clearly is injective with a compact domain, thus e^k is an embedding. This completes the induction argument and proves the existence of an embedding $e^k : \mathbb{T}^k \to B^{k+2}$.

Remark B.2. This argument is also true if we start with an embedding of $\mathbb{T}^1$ into B^2 so that we obtain an embedding of $\mathbb{T}^k$ into B^{k+1} for all $k \geqslant 1$. But we need one extra dimension to deal with the solid torus in what follows.

It is crucial to observe that the entire inductive construction just presented is built over nested tubular neighborhoods. Indeed, the image of e^{k+1} is contained in a tubular neighborhood of the image of e^k for each $k \geqslant 1$. In the above construction we consider the tubular neighborhood of e^k in $\mathbb{R}^{k+3}$. However since we proceed inductively using orthogonal bundles of the successive images of $e^{k+1}, e^{k+2}, \ldots,$ and we contract the diameter of the tubular neighborhood at each step by a constant factor $0 < \lambda < 1$, we have in fact that *the image of e^{k+l} is in a tubular neighborhood of e^k for all $k, l \geqslant 1$.*

Therefore, the distance between the image e^{k+l} and the image of e^k is bounded by $d \sum_{i=k}^{k+l} \lambda^i < d$, always *inside* a tubular neighborhood U^{k+l}_k of $e^k(\mathbb{T}^k)$ in $\mathbb{R}^{k+l+2}$. Hence there exists a projection $\pi^{k+l}_k : U^{k+l}_k \to e^k(\mathbb{T}^k)$ associated to this tubular neighborhood, for each $k \geqslant 1$ and $l \geqslant 0$.

B.2. The embedding of $\mathcal{T}^k$ into B^{k+2}

The previous discussion provides an embedding e^k of $\mathbb{T}^k$ into B^{k+2} for each $k \geqslant 1$. Therefore considering a tubular neighborhood U^k of the compact sub-manifold $e^k(\mathbb{T}^k)$ in B^{k+2} we obtain a projection $\pi : U \to e(\mathbb{T}^k)$ such that $\pi^{-1}(w)$ is a 2-disk. Thus we obtain an embedding $\hat{e}^k$ of $\mathbb{T}^k \times \mathbb{D}$ into B^{k+2}.

We can assume that the tubular neighborhood has radius smaller than $\lambda^k d$ and that $U^k \subset U^k_l$ for all $l < k$. Then we can consider the projections $\pi^k : U^k \to e^k(\mathbb{T}^k)$ and $\pi^k_l : U^k_l \to e^l(\mathbb{T}^l)$. In what follows we assume without loss of generality that U^k is the image of $\hat{e}^k$.
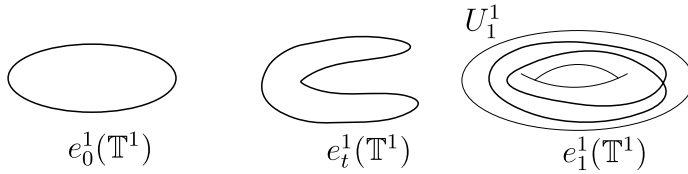
Fig. 14. The isotopy e_t^1 and the tubular neighborhood U_1^1 in the end.

B.3. The solenoid map through an isotopy of the identity

The previous construction of the embeddings e^k of $\mathbb{T}^k$ depends on the initial embedding e^1 of $\mathbb{T}^1$ on B^3. Moreover it is clear that each of the embeddings e^k for $k > 1$ depends smoothly on e^1. Hence a smooth family e_t^1 of embeddings, for $t \in [0, 1]$, defines a smooth family e_t^k of corresponding higher-dimensional embeddings.

We argue again by induction on $k \geqslant 1$. For $k = 1$ we consider the family of embeddings e_t^1 described in Fig. 14.

We can construct this family as depicted so that the extreme elements of the family satisfy $\pi_1^1 \circ e_1^1(\theta) = e_0^1(\theta^2)$ for $\theta \in \mathbb{T}^1$. If we choose a small tubular neighborhood of the image of e_t^1, then we obtain a family of embeddings $\hat{e}_t^1$ of $\mathcal{T}^1$ such that $(\pi_1^1 \circ \hat{e}_1^1)(\theta, z) = e_0^1(\theta^2)$.

Now we can use the family e_t^1 to construct families e_t^k of embeddings following the inductive procedure explained before, for each $t \in [0, 1]$ and each fixed $k \geqslant 1$.

Again taking a small tubular neighborhood of the image of e_t^k we obtain a family of embeddings $\hat{e}_t^k$ of $\mathcal{T}^k$ such that the image of $\hat{e}_1^k$ of $\mathcal{T}^k$ is compactly contained inside the image of $\hat{e}_0^k$.

Due to the nested construction, for each $k \geqslant 1$ we have $(\pi_1^k \circ \hat{e}^k)(\theta_1, \ldots, \theta_k, z) = e_0^1(\theta_1^2)$ after projecting into the lower-dimensional image. Moreover projecting on the previous stage of the construction we get $(\pi_{k-1}^k \circ \hat{e}^k)(\theta_1, \ldots, \theta_k, z) = e_0^{k-1}(\theta_2^2, \ldots, \theta_k^2) \cap (\pi_1^k)^{-1}\{e_0^1(\theta_1^2)\}$, which can easily be proved by induction on $k \geqslant 1$ following the nested construction presented above.

This is enough to prove that (independently of the definition of A_Θ in s)

$$\pi_k^k \circ \hat{e}_1^k = \pi_k^k \circ \hat{e}_0^k \circ s \quad \text{for all } k \geqslant 1. \tag{37}$$

Finally, since by definition $\hat{e}_1^k$ is a small tubular neighborhood of the image of e_1^k, and this set is contained inside a tubular neighborhood of the image of e_0^k, then from (37) we see that in fact there exists a family of contractions $(A_\Theta)_{\Theta \in \mathbb{T}^k}$ such that $\hat{e}_1^k = \hat{e}_0^k \circ s$. This completes the proof of Proposition B.1.

References

[1] V.S. Afraimovich, V.V. Bykov, L.P. Shil'nikov, On the appearance and structure of the Lorenz attractor, Dokl. Acad. Sci. USSR 234 (1977) 336–339.

[2] J.F. Alves, V. Araujo, Random perturbations of nonuniformly expanding maps, Astérisque 286 (2003) 25–62.

[3] J.F. Alves, V. Araújo, M.J. Pacifico, V. Pinheiro, On the volume of singular-hyperbolic sets, Dyn. Syst. 22 (3) (2007) 249–267.

[4] J.F. Alves, C. Bonatti, M. Viana, SRB measures for partially hyperbolic systems whose central direction is mostly expanding, Invent. Math. 140 (2) (2000) 351–398.

[5] J.F. Alves, S. Luzzatto, V. Pinheiro, Markov structures and decay of correlations for non-uniformly expanding dynamical systems, Ann. Inst. H. Poincaré Anal. Non Linéaire 22 (6) (2005) 817–839.

[6] J.F. Alves, M. Viana, Statistical stability for robust classes of maps with non-uniform expansion, Ergodic Theory Dynam. Systems 22 (2002) 1–32.

[7] V. Araújo, M.J. Pacifico, Three-Dimensional Flows, Ergeb. Math. Grenzgeb. (3) (Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics), vol. 53, Springer-Verlag, Heidelberg, 2010, with a foreword by Marcelo Viana.

[8] V. Araújo, E.R. Pujals, M.J. Pacifico, M. Viana, Singular-hyperbolic attractors are chaotic, Trans. Amer. Math. Soc. 361 (2009) 2431–2485.

 [9] C. Bonatti, A. Pumariño, M. Viana, Lorenz attractors with arbitrary expanding dimension, C. R. Acad. Sci. Paris Sér. I Math. 325 (8) (1997) 883–888.
[10] W. de Melo, S. van Strien, One-Dimensional Dynamics, Springer-Verlag, 1993.
[11] J.-M. Gambaudo, I. Procaccia, S. Thomae, C. Tresser, New universal scenarios for the onset of chaos in Lorenz-type flows, Phys. Rev. Lett. 57 (1986) 925–928.
[12] J.-M. Gambaudo, C. Tresser, On the dynamics of quasi-contractions, Bol. Soc. Brasil. Mat. 19 (1988) 61–114.
[13] J. Guckenheimer, R.F. Williams, Structural stability of Lorenz attractors, Publ. Math. Inst. Hautes Études Sci. 50 (1979) 59–72.
[14] M. Hirsch, C. Pugh, Stable manifolds and hyperbolic sets, in: Global Analysis, Berkeley, 1968, in: Proc. Sympos. Pure Math., vol. XIV, Amer. Math. Soc., 1970, pp. 133–163.
[15] M. Hirsch, C. Pugh, M. Shub, Invariant Manifolds, Lecture Notes in Math., vol. 583, Springer-Verlag, New York, 1977.
[16] D. Lyubimov, M.A. Zaks, Two mechanisms of the transition to chaos in finite-dimensional models of convection, Phys. D 9 (1983) 52–64.
[17] R. Mañé, Hyperbolicity, sinks and measure in one-dimensional dynamics, Comm. Math. Phys. 100 (1985) 495–524; Errata in: Comm. Math. Phys. 112 (1987) 721–724.
[18] R. Mañé, Ergodic Theory and Differentiable Dynamics, Springer-Verlag, New York, 1987.
[19] M. Martens, W. de Melo, Universal models for Lorenz maps, Ergodic Theory Dynam. Systems 21 (3) (2001) 833–860.
[20] R. Metzger, C. Morales, Sectional-hyperbolic systems, Ergodic Theory Dynam. Systems 28 (2008) 1587–1597.
[21] C.A. Morales, M.J. Pacifico, E.R. Pujals, Robust transitive singular sets for 3-flows are partially hyperbolic attractors or repellers, Ann. of Math. (2) 160 (2) (2004) 375–432.
[22] C.A. Morales, M.J. Pacifico, B. San Martin, Expanding Lorenz attractors through resonant double homoclinic loops, SIAM J. Math. Anal. 36 (6) (2005) 1836–1861.
[23] C.A. Morales, M.J. Pacifico, B. San Martin, Contracting Lorenz attractors through resonant double homoclinic loops, SIAM J. Math. Anal. 38 (1) (2006) 309–332.
[24] M.J. Pacifico, A. Rovella, M. Viana, Infinite-modal maps with global chaotic behavior, Ann. of Math. (2) 148 (2) (1998) 441–484; Corrigendum in: Ann. of Math. (2) 149 (1999) 705.
[25] M.M. Peixoto, On structural stability, Ann. of Math. (2) 69 (1959) 199–222.
[26] M.M. Peixoto, Structural stability on two-dimensional manifolds, Topology 1 (1962) 101–120.
[27] V. Pinheiro, Sinai–Ruelle–Bowen measures for weakly expanding maps, Nonlinearity 19 (5) (2006) 1185–1200.
[28] C. Robinson, Homoclinic bifurcation to a transitive attractor of Lorenz type, Nonlinearity 2 (4) (1989) 495–518.
[29] C. Robinson, Homoclinic bifurcation to a transitive attractor of Lorenz type. II, SIAM J. Math. Anal. 23 (5) (1992) 1255–1268.
[30] C. Robinson, Nonsymmetric Lorenz attractors from a homoclinic bifurcation, SIAM J. Math. Anal. 32 (1) (2000) 119–141.
[31] A. Rovella, The dynamics of perturbations of the contracting Lorenz attractor, Bull. Brazil. Math. Soc. 24 (2) (1993) 233–259.
[32] A.A. Simonov, An investigation of bifurcations in some dynamical systems by the methods of symbolic dynamics, Soviet Math. Dokl. 19 (1978) 759–763.
[33] A.A. Simonov, Investigation of piecewise monotone transformations of an interval by the methods of symbolic dynamics, Soviet Math. Dokl. 19 (1978) 185–188.
[34] S. Smale, Differentiable dynamical systems, Bull. Amer. Math. Soc. 73 (1967) 747–817.
[35] D.V. Turaev, L.P. Shil'nikov, An example of a wild strange attractor, Mat. Sb. 189 (2) (1998) 137–160.
[36] D.V. Turaev, L.P. Shilnikov, On bifurcations of a homoclinic "figure eight" for a saddle with negative saddle value, Soviet Math. Dokl. 34 (1987) 397–401.
[37] M. Viana, Multidimensional nonhyperbolic attractors, Publ. Math. Inst. Hautes Études Sci. 85 (1997) 63–96.
[38] M. Viana, Stochastic dynamics of deterministic systems, in: XXI Coloquio Brasileiro de Matematica [21th Brazilian Mathematics Colloquium], in: Publ. Mat. IMPA (IMPA Mathematical Publications), Instituto de Matemática Pura e Aplicada (IMPA), Rio de Janeiro, 1997.
[39] M. Viana, What's new on Lorenz strange attractor, Math. Intelligencer 22 (3) (2000) 6–19.