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Interpolation from number states to chaotic states of the electromagnetic field

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Abstract

We introduce new two-parameter states of the quantized radiation field interpolating from number states to chaotic states. Instead of dealing with pure states, we consider truncated mixtures of number states—number-chaotic states (NCS)—which reduce to number and chaotic (thermal) states in two well-defined limits. We study the statistical and squeezing properties of such states and show that there is a value of the chaoticity parameter $\bar{n}$ at which a transition from sub- to super-Poissonian characteristics occurs. Analysing the atomic population inversion in the Jaynes–Cummings model for these NCSs, we demonstrate the appearance of collapses and revivals as $\bar{n}$ is increased. Their phase space representations are studied emphasizing the changes as the parameters are varied and we discuss the nonclassical depth.

Keywords: Two-parameter states, quantized radiation field, electromagnetic field

(Some figures in this article are in colour only in the electronic version; see www.iop.org)

1. Introduction

The basic pure states one uses to deal with the properties of a single mode of the quantized radiation field are the number states [1], the coherent states [2], the squeezed states [3] and the phase states [4]. Superpositions of these states have been extensively studied as, for example, two number [5], coherent [6] and squeezed [7] states, as well as, discrete and continuous superpositions of coherent states on a line and on a circle [8–10]. Mixtures of number states, such as the chaotic (thermal) state [11] and the Poissonian mixed state [12], also play an important role when discussing the properties of the electromagnetic field. A seminal example is the superposition of a pure (coherent) state with a Gaussian (thermal) state (suggested by Glauber [13]) to describe the pollution of the state by thermal noise: this has been experimentally verified by Arecchi *et al* [14].

Other interesting examples among the large number of states studied in quantum optics are interpolating states which vary between two given states. To our knowledge, the pioneer of these states is the binomial state, introduced by Stoler *et al* [15], which goes from a number state to a coherent state. Other examples of such states are: the generalized geometric state (GGS), introduced by Obada *et al* [16], which interpolates between a number state and a chaotic state; the intermediate number-phase state [17], interpolating between a number state and the (Pegg–Barnett) phase state; the intermediate number-squeezed state [18], going from a number state to a squeezed state; the generalized superposition of coherent states [19], which varies between two arbitrary coherent states, including its extension for two squeezed states [20], and so on.

This paper is concerned with a new two-parameter state interpolating between a number state and a chaotic state of the electromagnetic field, which will be referred to hereafter as the number-chaotic state (NCS). This state is an alternative to the GGS presented in [16]; instead of pure states, which require some randomization to lose their purity and become chaotic, we consider truncated mixed states reducing to a number and a chaotic (thermal) state of the field in two well-defined limits. These states present interesting properties, linked with the nature of the photon statistics changing from sub- to super-Poissonian and collapses and revivals of the atomic population inversion, within the Jaynes–Cummings model, enhancing as the parameter measuring the degree of

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chaoticity is increased. The paper is organized as follows: in the section 2, we discuss the interpolation between number and chaotic states, presenting a brief summary of the GGS, and introduce our proposal for the NCS. In section 3, we study nonclassical properties (the sub-Poissonian effect, squeezing, atomic population inversion) of this interpolating state while section 4 is devoted to its phase space representations and to the discussion of the nonclassical depth. Section 5 contains our conclusions.

2. Interpolating from number to chaotic state

The density operator corresponding to a chaotic (thermal) state is given by

$$\hat{\rho}_{\mathrm{Ch}}(\bar{n}) = \sum_{m=0}^{\infty} P_m^B(\bar{n})\ |m\rangle\ \langle m| \tag{2.1}$$

where $P_m^B(\bar{n})$ is the Bose–Einstein (geometric) distribution given by

$$P_m^B(\bar{n}) = \frac{1}{1+\bar{n}}\left(\frac{\bar{n}}{1+\bar{n}}\right)^m \tag{2.2}$$

with the parameter $\bar{n}$, which specifies the state, representing the mean number of photons, $\langle\hat{n}\rangle = \bar{n}$; for a thermal state it becomes

$$\bar{n} = \left(\mathrm{e}^{\hbar\omega/kT} - 1\right)^{-1} \tag{2.3}$$

ω being the field frequency and T representing the temperature of the radiation source. Note that if one defines

$$A = \frac{\bar{n}}{1+\bar{n}} < 1 \tag{2.4}$$

($A = \exp(-\hbar\omega/kT)$ for a thermal state), the Bose–Einstein distribution can be rewritten as

$$P_m^B(A) = (1-A)\,A^m. \tag{2.5}$$

As it is well known, if one makes $\bar{n} \to 0$ ($A \to 0$), which corresponds to taking the temperature to zero in the case of thermal states, then

$$\lim_{\bar{n}\to 0} P_m^B(\bar{n}) = \delta_{m,0}, \tag{2.6}$$

so that the thermal state reduces to the vacuum state in this limit. In the reverse way, one might say that the thermalization of the vacuum leads to a thermal state; this is precisely the view point of the thermofield dynamics formalism with the introduction of thermal vacua [21]. One can naturally think about the thermalization of a given Fock state $|N\rangle$ and question whether an intermediate state can be constructed interpolating between a number and a chaotic (thermal) state. Initially, we comment on the proposal by Obada *et al* [16], known as the GGS, and then present an alternative to it which we call the NCS.

2.1. The generalized geometric state (GGS)

This interesting state was introduced in [16], its main characteristics being summarized here. It is a three-parameter interpolating state, denoted by $|y, N\rangle$ (where y is a complex number), which is specified by

$$|y, N\rangle = \sum_{n=0}^{N} C_n^N(y)\,|n\rangle, \tag{2.7}$$

where

$$C_n^N(y) = \lambda y^{n/2}, \tag{2.8}$$

with $y = |y|\mathrm{e}^{2\mathrm{i}\psi}$ (restricted to $|y| \neq 1$) and $\lambda^2 = (1-|y|)/(1-|y|^{N+1})$.

This definition yields an interpolation between the number and chaotic states in the sense that these states appear as the two limiting cases described below.

(a) *Number-state limit.* The GGS can be mapped onto a number state $|N\rangle$ by setting $\psi = 0$ and taking the limit $|y| \to \infty$ since, in this case,

$$C_n^N(y) \xrightarrow{|y|\gg 1} |y|^{(n-N)/2} \xrightarrow{|y|\to\infty} \delta_{n,N} \tag{2.9}$$

for all $n \leqslant N$.

(b) *Chaotic-state limit.* Now, taking $|y| = \bar{n}/(1+\bar{n}) < 1$ and $N \to \infty$, the density operator representing the field in this limit reads

$$\begin{aligned}\hat{\rho}\ (y,\infty) &= |y,\infty\rangle\ \langle y,\infty| \\ &= \lambda^2 \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} |y|^{(n-m)/2}\mathrm{e}^{2\mathrm{i}\psi(n-m)}\ |n\rangle\ \langle m|\,.\end{aligned} \tag{2.10}$$

Assuming that ψ is a random phase and implementing an ensemble average upon the state in (2.10), one obtains

$$\begin{aligned}\langle\hat{\rho}\ (y,\infty)\rangle_\psi &= \frac{1}{2\pi}\int_0^{2\pi} \hat{\rho}\ (y,\infty)\,\mathrm{d}\psi \\ &= \sum_{m=0}^{\infty} \frac{\bar{n}^m}{(1+\bar{n})^{m+1}}\ |m\rangle\ \langle m|\,.\end{aligned} \tag{2.11}$$

One readily recognizes in (2.11) the Bose distribution and hence, with the parameter choices $|y|$, N and ψ, the GGS is mapped onto a chaotic (or thermal) state. It should be emphasized that to get a chaotic state from a pure state, as is the case of the GGS, one has to perform randomization by hand, which in the present case corresponds to an average over the phase of the parameter y. Nonclassical properties and phase space representations of the GGS were studied in detail in [16].

2.2. The number-chaotic state (NCS)

A distinct method, permitting interpolation between number and chaotic states, is to rely on the observation made in the beginning of this section and to start from a mixture of number states, with weights somehow preserving the characteristics of the Bose distribution, looking for chaotic mixed states which become a pure number state when the chaotic nature is suppressed. Since the thermal (chaotic) state reduces to the vacuo state $|0\rangle$ when the temperature goes to zero, it seems reasonable that to interpolate between a number state $|N\rangle$ and a chaotic state one should consider truncated mixtures of number states including only Fock states $|m\rangle$ with $m \geqslant N$.

Here we introduce a new interpolating state, the NCS, which constitutes an alternative to the GGS as the truncated mixed state defined by

$$\hat{\rho}_{\rm NCS}(\bar{n}, N) = \frac{1}{(1+\bar{n})^N} \sum_{m=0}^{\infty} \frac{(N+m)!}{N!\,m!} P_m^B(\bar{n})\, |N+m\rangle\, \langle N+m| \tag{2.12}$$

where $P_m^B(\bar{n})$ is the Bose–Einstein distribution given by (2.2). Note that the mixed state (2.12) is properly normalized as a consequence of the identity [22]

$$\sum_{m=0}^{\infty} \frac{(N+m)!}{N!\,m!} A^m = \frac{1}{(1-A)^{N+1}} \tag{2.13}$$

which holds for $|A| < 1$.

This two-parameter state reduces to the number state $|N\rangle$ and to the thermal state in the following two well-defined limits.

(a) *The number-state limit.* The NCS tends to a number state when the parameter $\bar{n}$ tends to zero. In fact, $P_m^B(\bar{n}) \to \delta_{m,0}$ and this allows us to write the statistical distribution of the NCS, in this limit, as

$$p_{N+m}(\bar{n}, N) = \frac{1}{(1+\bar{n})^N} \frac{(N+m)!}{N!\,m!} P_m^B(\bar{n}) \overset{\bar{n}\to 0}{\longrightarrow} \delta_{m,0} \tag{2.14}$$

and we find from (2.12) that

$$\hat{\rho}_{\rm NCS}(\bar{n} \to 0, N) = |N\rangle\, \langle N|; \tag{2.15}$$

in other words, when the parameter $\bar{n}$ (which may be seen as measuring the degree of chaoticity) tends to zero one recovers the pure number state $|N\rangle$.

(b) *The chaotic-state limit.* On the other hand, it follows immediately that $p_{N+m}(\bar{n}, 0) = P_m^B(\bar{n})$ and from (2.12) we conclude that the NCS, for $N = 0$, becomes the chaotic state with a mean number of photons equal to $\bar{n}$, that is

$$\hat{\rho}_{\rm NCS}(\bar{n}, N = 0) = \hat{\rho}_{\rm Ch}(\bar{n}). \tag{2.16}$$

These two limits of the NCS show that it constitutes an interpolation between number and chaotic states, constructed from a viewpoint rather distinct from the perspective of [16].

It should be noted that (2.12) is not the unique truncated mixture of number states that possesses the limits mentioned above. For instance, by suppressing the combinatorial factor in the expression (2.12), and properly adjusting the normalization constant ($\mathcal{N} = 1$), one would still have a truncated mixed state interpolating between the number and chaotic states; such a state was introduced by Lee [23] (named the shifted thermal state) inspired by the work of Scully *et al* [24]. Actually, the factor $P_m^B(\bar{n})$ must appear in the weight of the state $|N\rangle$ in the mixture in order to guarantee (since $\lim_{\bar{n}\to 0} P_m^B(\bar{n}) = \delta_{m,0}$) that the mixture reduces to the state $|N\rangle$ alone as $\bar{n}$ goes to zero; but any prescription for the statistical weights of the mixture in the form $p_{N+m}(\bar{n}, N) = \mathcal{N} f_m(\bar{n}, N) P_m^B(\bar{n})$ will lead to the same limits if $f_m(\bar{n}, 0) = 1$ and the sequence of values $f_m(\bar{n} \to 0, N)$ is limited, with f_m going quickly to 1 as m increases. The choice made here somehow incorporates the fact that, for a given N, the lowest energy states from $|0\rangle$ to $|N-1\rangle$ are not populated, that is, they do not participate in the truncated mixture; it corresponds to extend the thermalization of the vacuum state to a number state. In fact, equation (2.12) can be rewritten in the form

$$\hat{\rho}_{\rm NCS}(\bar{n}, N) = \frac{1}{(1+\bar{n})^N} \sum_{m=0}^{\infty} P_m^B(\bar{n}) \frac{1}{m!} \left(\hat{a}^{\dagger m}|N\rangle\right)\left(\hat{a}^{\dagger m}|N\rangle\right)^{\dagger}, \tag{2.17}$$

which becomes the density matrix of the chaotic (thermal) state if one replaces $|N\rangle$ by the vacuum state $|0\rangle$; in other words, the NCS has been built up as a mimic of the thermal state and it can therefore be viewed as a kind of thermalization of the number state $|N\rangle$. One could naturally try to incorporate the NCS within the context of the thermofield dynamics formalism [21] but we will not pursue this issue here.

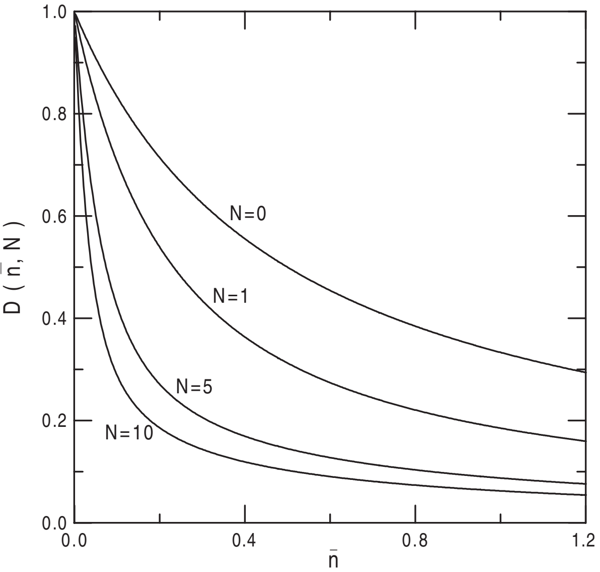


Figure 1. Trace of the square of the density matrix for the NCS, $D(\bar{n}, N)$, plotted as a function of $\bar{n}$ for various values of N.

The probability distribution of photons of the NCS, $p_{N+m}(\bar{n}, N)$, thus resembles the one of the shifted thermal state, being modulated by the combinatorial factor. As the NCS is a mixed state, one can measure its departure from purity by calculating the trace of the square of its density matrix, which indicates the deviation from the idempotent property:

$$\mathrm{D}(\bar{n}, N) = \mathrm{Tr}[\hat{\rho}_{\rm NCS}^2] = \frac{1}{(1+\bar{n})^{2N}} \sum_{m=0}^{\infty} \left[\frac{(N+m)!}{N!\,m!} P_m^B(\bar{n})\right]^2. \tag{2.18}$$

Figure 1 illustrates the behaviour of $\mathrm{D}(\bar{n}, N)$ as a function of $\bar{n}$ for some values of N; one sees that the NCS loses its purity, as $\bar{n}$ increases, more quickly for large N than for small N, that is, Fock states with a small number of photons are more resistant to degradation (by introducing chaoticity in the way it is done here) than more intense fields, an aspect similar to thermal decoherence effects [25].

3. Nonclassical properties of the NCS

We now discuss some nonclassical properties of the NCS focussing on the variations in these aspects as we interpolate between the number and chaotic states.

3.1. Antibunching and sub-Poissonian effects

These two nonclassical effects occur simultaneously when we are concerned with a stationary single-mode field, as in the present case. Consequently, the study of one of them is enough to establish the values of the system parameters for which these effects are relevant. Sub-Poissonian statistics occurs when the dispersion in the number operator $\hat{n}$,

$$\Delta\hat{n}^2 = \langle\hat{n}^2\rangle - \langle\hat{n}\rangle^2 \tag{3.1}$$

is smaller than the mean number of photons $\langle\hat{n}\rangle$, that is, when the Mandel factor Q, defined by [26]

$$Q = (\Delta\hat{n}^2 - \langle\hat{n}\rangle)/\langle\hat{n}\rangle, \tag{3.2}$$

is negative, Q belonging to the interval $[-1, 0)$; otherwise, the state is said to be Poissonian, when $Q = 0$, or super-Poissonian if $Q > 0$.

Both $\langle\hat{n}\rangle$ and $\langle\hat{n}^2\rangle$ can be easily calculated using (2.4), (2.5) and (2.13). In fact, since

$$\sum_{m=0}^{\infty}\frac{(N+m)!}{N!\,m!}A^m m = (N+1)\frac{A}{(1-A)^{N+2}}, \tag{3.3}$$

$$\begin{aligned}\sum_{m=0}^{\infty}\frac{(N+m)!}{N!\,m!}A^m m^2 &= (N+1)(N+2)\frac{A^2}{(1-A)^{N+3}}\\ &+(N+1)\frac{A}{(1-A)^{N+2}},\end{aligned} \tag{3.4}$$

one finds

$$\langle\hat{n}\rangle_{\rm NCS} = \frac{1}{(1+\bar{n})^N}\sum_{m=0}^{\infty}\frac{(N+m)!}{N!\,m!}P_m^B(\bar{n})\,(N+m) = N + (N+1)\,\bar{n} \tag{3.5}$$

and

$$\begin{aligned}\langle\hat{n}^2\rangle_{\rm NCS} &= \frac{1}{(1+\bar{n})^N}\sum_{m=0}^{\infty}\frac{(N+m)!}{N!\,m!}P_m^B(\bar{n})\,(N+m)^2\\ &= N^2 + (2N+1)(N+1)\,\bar{n} + (N+1)(N+2)\,\bar{n}^2,\end{aligned} \tag{3.6}$$

where $\bar{n}$ (the chaoticity parameter) is the mean number of photons in the chaotic state $\hat{\rho}_{\rm Ch}(\bar{n})$. As one would expect, the mean number of photons of a NCS is given by N plus a contribution linear in $\bar{n}$. The substitution of (3.5) and (3.6) into (3.1) and (3.2) gives the Mandel factor of the NCS,

$$Q(\bar{n}, N) = \frac{(N+1)\,\bar{n}^2 - N}{(N+1)\,\bar{n} + N}. \tag{3.7}$$

This Mandel parameter for the NCS naturally reduces to those of the number state $|N\rangle$ and of the chaotic state in the appropriated limits, that is,

$$\begin{aligned}Q(\bar{n}\to 0, N) &= Q_{|N\rangle} = -1 + \delta_{N,0},\\ Q(\bar{n}, N=0) &= Q_{\rm Ch}(\bar{n}) = \bar{n};\end{aligned} \tag{3.8}$$

$Q(\bar{n}, N)$ is plotted in figure 2 as a function of $\bar{n}$ for some values of N. From this plot, and directly from equation (3.7), one sees that there is a critical value of $\bar{n}$ for which a transition from sub-Poissonian to super-Poissonian characteristics occurs; this critical value is

$$\bar{n}_c^{(N)} = \sqrt{\frac{N}{N+1}}, \tag{3.9}$$

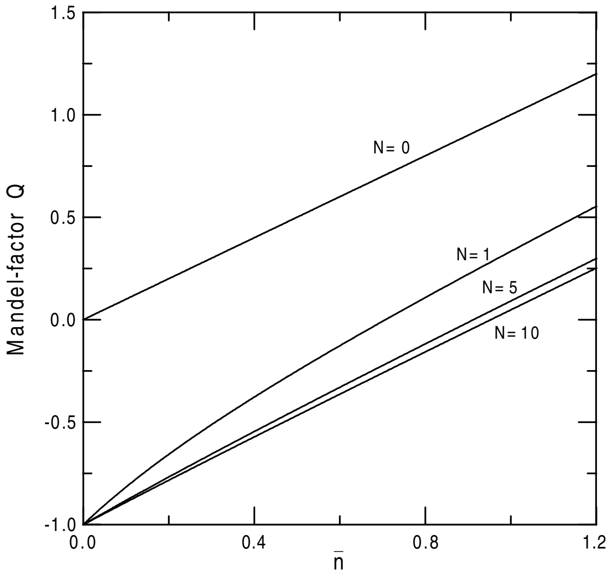


Figure 2. Mandel parameter for some NCS as a function of the chaoticity parameter $\bar{n}$; the state specified by $N = 0$ corresponds to the chaotic (thermal) state.

the NCS having $Q < 0$ below it and $Q > 0$ above. With exception of the case $N = 0$, the chaotic state (which can be seen as the NCS associated to $|0\rangle$), one has $1/\sqrt{2} \leqslant \bar{n}_c^{(N)} < 1$ and so all NCS (with $N > 1$) are sub-Poissonian for low value of $\bar{n}$ ($< 1/\sqrt{2}$) and super-Poissonian when $\bar{n} > 1$.

3.2. Squeezing effect

To obtain the dispersion

$$\Delta\hat{x}_l^2 = \langle\hat{x}_l^2\rangle - \langle\hat{x}_l\rangle^2 \tag{3.10}$$

in the field-quadrature operators

$$\hat{x}_l = \frac{i^{1-l}}{2}\big(\hat{a} - (-1)^l\hat{a}^\dagger\big), \qquad l = 1, 2, \tag{3.11}$$

which satisfy $[\hat{x}_1, \hat{x}_2] = {\rm i}/2$, we first calculate

$$\begin{aligned}\langle\hat{x}_l\rangle_{\rm NCS} &= {\rm Tr}[\hat{\rho}_{\rm NCS}\,\hat{x}_l] = \frac{1}{(1+\bar{n})^N}\sum_{m=0}^{\infty}\frac{(N+m)!}{N!\,m!}P_m^B(\bar{n})\\ &\times\langle N+m\,|\,\hat{x}_l\,|\,N+m\rangle = 0\end{aligned} \tag{3.12}$$

and, similarly

$$\begin{aligned}\langle\hat{x}_l^2\rangle_{\rm NCS} &= \frac{1}{(1+\bar{n})^N}\sum_{m=0}^{\infty}\frac{(N+m)!}{N!\,m!}P_m^B(\bar{n})\,\langle N+m\,|\,\hat{x}_l^2\,|\,N+m\rangle\\ &= \frac{1}{4(1+\bar{n})^N}\sum_{m=0}^{\infty}\frac{(N+m)!}{N!\,m!}P_m^B(\bar{n})\\ &\times\langle N+m|(2\hat{a}^\dagger\hat{a}+1)|\,N+m\rangle\\ &= \frac{2\langle\hat{n}\rangle + 1}{4}.\end{aligned} \tag{3.13}$$

The substitution of (3.12) and (3.13) in (3.10) shows that the dispersion in both quadratures of the NCS are equal,

$$\Delta\hat{x}_l^2 = \frac{2\langle\hat{n}\rangle + 1}{4}. \tag{3.14}$$

The above result, expressing the square of the quadrature variances in terms of the mean number of photons, actually holds for all mixtures of number states [12] (including thermal states, for which it becomes $\Delta\hat{x}_l^2 = (2\bar{n}+1)/4$) and is formally identical to that of a single number state $|N\rangle$, for which $\Delta\hat{x}_i^2 = (2N+1)/4$. For the NCS, using (3.5), one finds

$$(\Delta\hat{x}_l^2)_{\mathrm{NCS}} = \tfrac{1}{4}\{2N+1+2(N+1)\,\bar{n}\}\,. \tag{3.15}$$

Since the squeezing effect would occur when $\Delta x_l^2 < 1/4$, for either $l = 1$ or 2 (but not both simultaneously), one sees that the NCS do not exhibit quadrature squeezing; the equality $\Delta\hat{x}_2^2 = \Delta\hat{x}_1^2$ plus the squeezing condition would imply violation of the Heisenberg inequality ($\Delta\hat{x}_1^2\Delta\hat{x}_2^2 \geqslant 1/16$). This result does not come from the fact that the NCS interpolates between two states showing no squeezing, the GGS being a counter-example (other examples come from the interpolating states studied in [5] and [17]); it is only due to the fact that the NCS is a mixed state.

3.3. Atomic inversion

In experiments on electromagnetic cavities one monitors the population of atomic states as a function of time. For the case of a two-level (Rydberg) atom interacting with the field as described by the single-photon Jaynes–Cummings model in the rotating wave approximation, the Hamiltonian is given by

$$\hat{H} = \frac{\hbar\omega_0}{2}\,\hat{\sigma}_3 + \hbar\omega\,\hat{a}^\dagger\hat{a} + \hbar\lambda(\hat{\sigma}_+\,\hat{a} + \hat{\sigma}_-\,\hat{a}^\dagger), \tag{3.16}$$

where ω_0 is the frequency of the transition between the ground and the excited atomic states ($\hbar\omega_0 = E_{|\mathrm{e}\rangle} - E_{|\mathrm{g}\rangle}$), ω is the field-mode frequency and λ is the atom–field coupling parameter. In the above equation, $\hat{\sigma}_3$ is a Pauli matrix and $\hat{\sigma}_+$ and $\hat{\sigma}_-$ correspond to the raising and lowering operators in the atomic two-level basis, which are given by

$$\hat{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \hat{\sigma}_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \hat{\sigma}_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{3.17}$$

For a general state of the field, described by the density matrix $\hat{\rho}_{\mathrm{F}}$, the time evolution of the atomic population inversion is given by

$$\begin{aligned} W(t) &= \mathrm{Tr}\left[\hat{\rho}_{\mathrm{AF}}(t)\,\hat{\sigma}_3\right] \\ &= \mathrm{Tr}\left[\mathrm{e}^{-\mathrm{i}\hat{H}t/\hbar}\left\{|\psi_{\mathrm{A}}(0)\rangle\langle\psi_{\mathrm{A}}(0)|\otimes\hat{\rho}_{\mathrm{F}}(0)\right\}\mathrm{e}^{\mathrm{i}\hat{H}t/\hbar}\,\hat{\sigma}_3\right], \end{aligned} \tag{3.18}$$

where $|\psi_{\mathrm{A}}\rangle$ represents a pure atomic state. When the initial atomic state is a coherent superposition of the ground and excited states

$$|\psi_{\mathrm{A}}(0)\rangle = c_{\mathrm{g}}\,|\mathrm{g}\rangle + c_{\mathrm{e}}\,|\mathrm{e}\rangle\,, \tag{3.19}$$

the atomic population inversion is given by [12, 27]

$$\begin{aligned} W_\delta(t) &= |c_{\mathrm{e}}|^2\left\{1 - 2\sum_{l=0}^{\infty}\frac{l+1}{\nu+1}\sin^2\left(\lambda t\sqrt{\nu+1}\right)\langle l|\,\hat{\rho}_{\mathrm{F}}(0)|l\rangle\right\} \\ &\quad -|c_{\mathrm{g}}|^2\left\{1 - 2\sum_{l=0}^{\infty}\frac{l}{\nu}\sin^2(\lambda t\sqrt{\nu})\langle l|\hat{\rho}_{\mathrm{F}}(0)|l\rangle\right\} \\ &\quad +2\,|c_{\mathrm{e}}|\,|c_{\mathrm{g}}|\sum_{l=0}^{\infty}\sqrt{l+1}\left[\sin(\phi+\gamma)\,\frac{\sin(2\lambda t\sqrt{\nu+1})}{\sqrt{\nu+1}}\right. \\ &\quad \left. +\delta\,\cos(\phi+\gamma)\,\frac{\sin^2(\lambda t\sqrt{\nu+1})}{\nu+1}\right]|\langle l|\hat{\rho}_{\mathrm{F}}(0)|l+1\rangle| \end{aligned} \tag{3.20}$$

where $\nu = l + \delta^2/4$, $\delta = (\omega_0 - \omega)\lambda^{-1}$ is the detuning parameter, with ϕ and γ defined by $c_{\mathrm{e}}\,c_{\mathrm{g}}^* = |c_{\mathrm{e}}|\,|c_{\mathrm{g}}|\,\exp(-\mathrm{i}\phi)$ and $\langle l|\hat{\rho}_{\mathrm{F}}(0)|l+1\rangle = |\langle l|\hat{\rho}_{\mathrm{F}}(0)|l+1\rangle|\exp(-\mathrm{i}\gamma)$. One should note that the last term in the above expression does not appear when one deals with a mixture of number states but it is relevant for treating pure states; in fact, for pure states expression (3.20) reduces to the expression obtained by Chaba *et al* [28].

Considering the resonant case ($\delta = 0$) and the initial state of the atom being the excited state $|\mathrm{e}\rangle$, equation (3.20) becomes

$$W_0(t) = \sum_{l=0}^{\infty}\cos\left(2\lambda t\sqrt{l+1}\right)\langle l|\,\hat{\rho}_{\mathrm{F}}(0)\,|l\rangle \tag{3.21}$$

and the atomic inversion in this case, when the field is initially in the NCS, is given by

$$\begin{aligned} W_0^{(\mathrm{NCS})}(t) &= \frac{1}{(1+\bar{n})^N}\sum_{m=0}^{\infty}\frac{(N+m)!}{N!\,m!}P_m^B(\bar{n}) \\ &\quad \times\cos\left(2\lambda t\sqrt{N+m+1}\right). \end{aligned} \tag{3.22}$$

For $\bar{n} \to 0$, this expression reduces to the inversion for the number state $|N\rangle$, $W_0^{|N\rangle}(t) = \cos\left(2\lambda t\sqrt{N+1}\right)$, whereas for $N = 0$ and $\bar{n} \neq 0$ one obtains the atomic inversion for a chaotic state

$$W_0^{(\mathrm{Ch})}(t) = \sum_{m=0}^{\infty}P_m^B(\bar{n})\cos\left(2\lambda t\sqrt{m+1}\right). \tag{3.23}$$

The atomic inversion, in the case of the NCS with $N = 1$, is illustrated in figure 3 for some values of $\bar{n}$. It is seen that, from a nearly regular oscillatory behaviour at low $\bar{n}$ (similar to that of the number state $|1\rangle$), one gets a more chaotic inversion pattern, like that of the thermal ($N = 0$) state [29], as $\bar{n}$ increases. For larger N, however, the behaviour is distinct: collapses and revivals of the atomic population inversion appear and are enhanced by raising $\bar{n}$. This interesting feature of the NCS is illustrated in figure 4 where the atomic inversion is plotted as a function of the rescaled time λt for $N = 5$ and some values of $\bar{n}$.

It follows from the discussion of the nature of the photon statistics and of the atomic inversion made above that the nonclassical properties of the NCS gradually (and quickly) disappear as $\bar{n}$ is increased; it looks as if the NCS evolves continuously from being quantum to becoming classical as the chaoticity parameter is raised from zero. To better analyse this point one should investigate their representations in phase space for increasing values of $\bar{n}$.

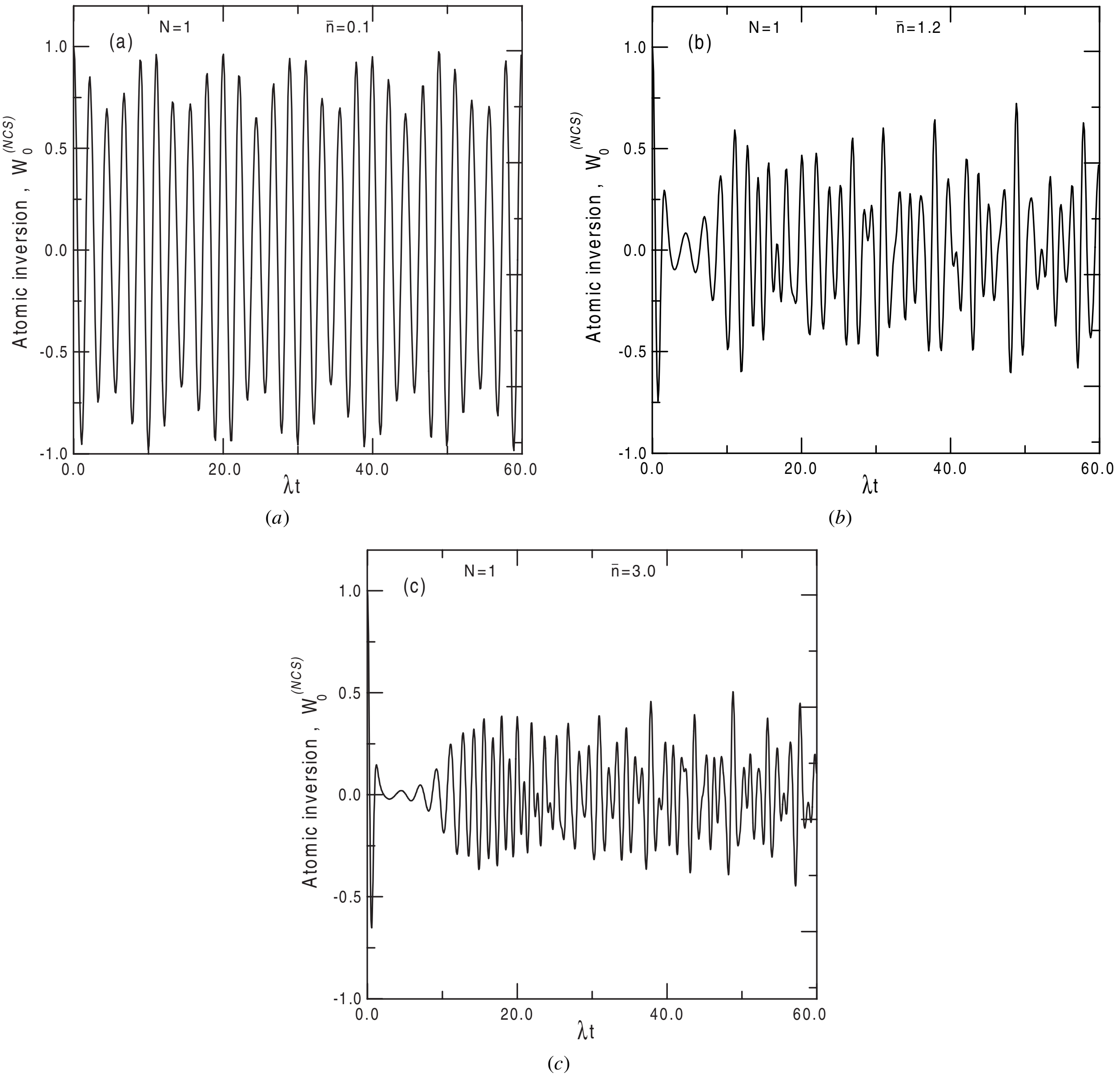


Figure 3. Atomic population inversion $W_0^{(\mathrm{NCS})}$, as a function of λt, for the field initially prepared in the NCS with $N = 1$, at various values of $\bar{n}$: (*a*) $\bar{n} = 0.1$; (*b*) $\bar{n} = 1.2$; (*c*) $\bar{n} = 3.0$.

4. Phase space representations of NCS

The basic representations of a field state in phase space are the P-, Q- and Wigner functions, which are linear representations of the corresponding density matrix, defined as two-dimensional Fourier transforms of the normally, antinormally and symmetrically ordered characteristic functions, $\chi_N(\eta) = \mathrm{Tr}[\hat{\rho}\,\mathrm{e}^{\eta\hat{a}^\dagger}\mathrm{e}^{-\eta^*\hat{a}}]$, $\chi_A(\eta) = \mathrm{Tr}[\hat{\rho}\,\mathrm{e}^{-\eta^*\hat{a}}\mathrm{e}^{\eta\hat{a}^\dagger}]$ and $\chi(\eta) = \mathrm{Tr}[\hat{\rho}\,\mathrm{e}^{\eta\hat{a}^\dagger-\eta^*\hat{a}}]$ respectively:

$$P(\beta) = \frac{1}{\pi^2}\int \mathrm{d}^2\eta\, \exp\left(\beta\eta^* - \beta^*\eta\right)\, \chi_N(\eta) \tag{4.1}$$

$$Q(\beta) = \frac{1}{\pi^2}\int \mathrm{d}^2\eta\, \exp\left(\beta\eta^* - \beta^*\eta\right)\, \chi_A(\eta) \tag{4.2}$$

$$W(\beta) = \frac{1}{\pi^2}\int \mathrm{d}^2\eta\, \exp\left(\beta\eta^* - \beta^*\eta\right)\, \chi(\eta), \tag{4.3}$$

where β and η are complex and $\mathrm{d}^2\eta = \mathrm{d}(\mathrm{Re}\,\eta)\,\mathrm{d}(\mathrm{Im}\,\eta)$ [11]. The P representation, introduced by Glauber and Sudarshan [2], is the diagonal representation of the density operator in the coherent basis and can also be defined by

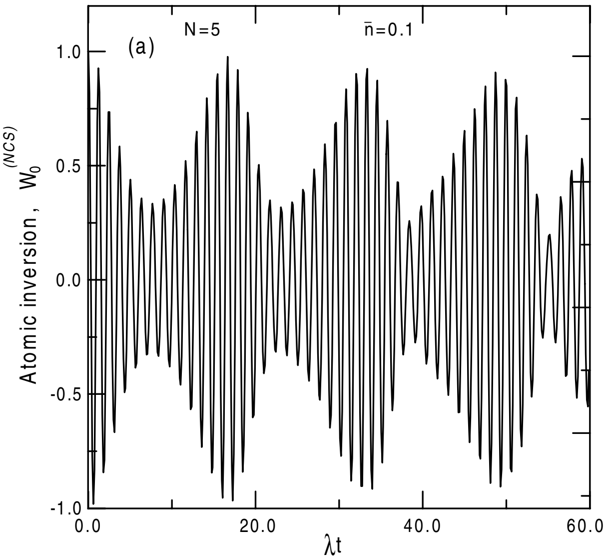
$$\hat{\rho} = \int \mathrm{d}^2\beta\, P(\beta)\, |\beta\rangle\,\langle\beta| \tag{4.4}$$

where $\beta = x + \mathrm{i}y$ and $\mathrm{d}^2\beta = \mathrm{d}x\,\mathrm{d}y$, while the Q-function corresponds to the diagonal matrix elements of $\hat{\rho}$ in the coherent basis, namely

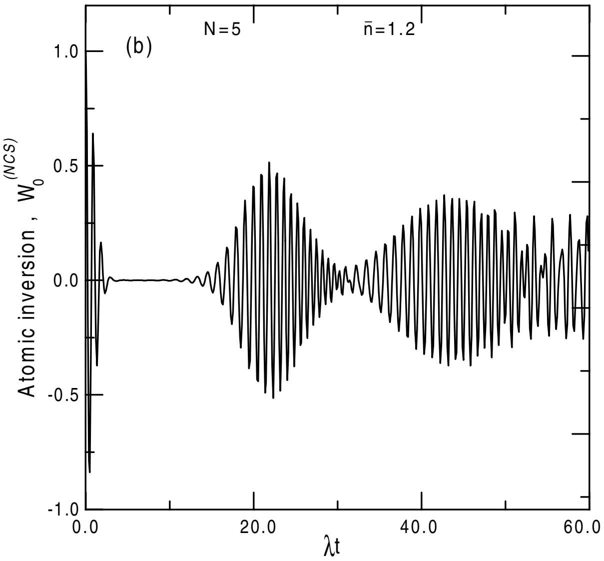
$$Q(\beta) = \frac{1}{\pi}\,\langle\beta|\,\hat{\rho}\,|\beta\rangle\,. \tag{4.5}$$

On the other hand, the Wigner function is a coordinate–momentum representation which can be introduced as

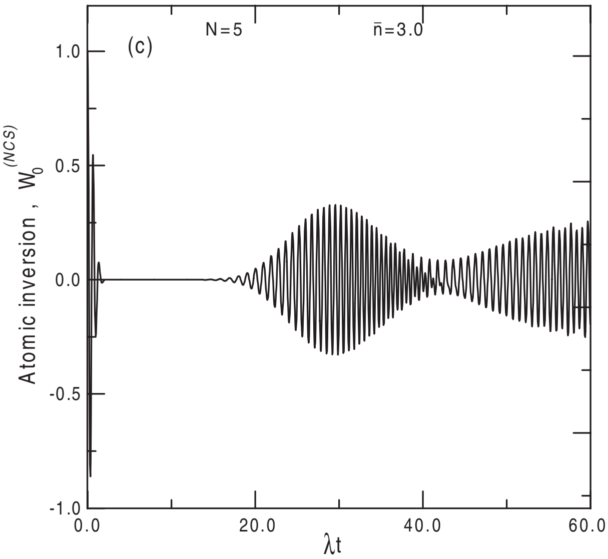
$$W(x, y) = \frac{1}{\pi}\int \langle x - z/2|\;\hat{\rho}\;|x + z/2\rangle \mathrm{e}^{-\mathrm{i}\,y\,z}\,\mathrm{d}z \tag{4.6}$$



(*a*)



(*b*)



(*c*)

Figure 4. Same as figure 3 but with $N = 5$: (*a*) $\bar{n} = 0.1$; (*b*) $\bar{n} = 1.2$; (*c*) $\bar{n} = 3.0$.

and, as such, cannot be a true probability distribution for a quantum state since, in this case, even being regular it possesses ranges of negative values. In contrast, the P-function is usually highly singular while the Q-function is always a positive regular function. Both the Q- and the Wigner functions are Gaussian convolutions of the P-function:

$$
\begin{aligned}
W(\beta) &= \frac{2}{\pi}\int \mathrm{d}^2\eta\, P(\eta)\,\mathrm{e}^{-2|\eta-\beta|^2}, \\
Q(\beta) &= \frac{1}{\pi}\int \mathrm{d}^2\eta\, P(\eta)\,\mathrm{e}^{-|\eta-\beta|^2},
\end{aligned}
\tag{4.7}
$$

which accounts for their rather better behaviour.

Consider an arbitrary mixed state given by

$$\hat{\rho}_{\mathrm{mix}} = \sum_{j=0}^{\infty} p_j |\psi_j\rangle\langle\psi_j| \tag{4.8}$$

with $\sum_{j=0}^{\infty} p_j = 1$, where $|\psi_j\rangle$ is a normalized pure state of the field and, in particular, the case where $|\psi_j\rangle = |j\rangle$ is a number state. The linearity of these representations assures that the functions representing the mixed state (4.8) can be expressed in terms of its constituents as

$$P_{\mathrm{mix}}(\beta) = \sum_{j=0}^{\infty} p_j\, P_{|\psi_j\rangle}(\beta) \tag{4.9}$$

$$Q_{\mathrm{mix}}(\beta) = \sum_{j=0}^{\infty} p_j\, Q_{|\psi_j\rangle}(\beta) \tag{4.10}$$

$$W_{\mathrm{mix}}(\beta) = \sum_{j=0}^{\infty} p_j\, W_{|\psi_j\rangle}(\beta)\ , \tag{4.11}$$

with $P_{|\psi_j\rangle}(\beta)$, $Q_{|\psi_j\rangle}(\beta)$ and $W_{|\psi_j\rangle}(\beta)$ denoting the P-, Q- and Wigner functions of the state $|\psi_j\rangle$, respectively. Since the

P-function is a more complicated object, we concentrate on the Q- and Wigner functions.

4.1. The Q-function of the NCS

For a single number state $|j\rangle$, the Q-function is given by

$$Q_{|j\rangle}(\beta) = \frac{1}{\pi}\,|\langle j \mid \beta\rangle|^2 = \frac{1}{\pi}\,\exp(-|\beta^2|)\,\frac{|\beta^2|^j}{j!} \qquad (4.12)$$

and therefore it follows from (2.12) and (4.10) that the Q-function of the NCS is given by

$$Q_{\rm NCS}(x, y) = \frac{1}{\pi}\,\exp(-x^2 - y^2) \times \frac{1}{(1+\bar{n})^N}\sum_{m=0}^{\infty}\frac{(N+m)!}{N!\,m!}\,P_m^B(\bar{n})\,\frac{(x^2+y^2)^{N+m}}{(N+m)!}. \qquad (4.13)$$

Note that, for $N = 0$, the above equation reduces to the Q-function of the chaotic state

$$Q_{\rm Ch}(x, y) = \frac{1}{\pi}\,\frac{1}{1+\bar{n}}\,\exp\left(-\frac{x^2+y^2}{1+\bar{n}}\right), \qquad (4.14)$$

which becomes the Gaussian Q-function of the vacuum state as $\bar{n} \to 0$; the temperature effect on the vacuum state is manifested in the broadening of this Gaussian function.

On the other hand, the Q-function of a number state $|N\rangle$ (with $N \neq 0$) is a Gaussian in which a crater is dug symmetrically, reaching zero at the origin. As expected, the Q-function of the NCS preserves this same form of a nonactive volcano, with the mountain becoming broader and losing its height, as $\bar{n}$ is increased. To investigate any eventual change in the quantum nature of the NCS as $\bar{n}$ is varied, one should study another phase-space representation, such as the Wigner quasi-probability distribution, of such states.

4.2. Wigner function of the NCS

Similarly, the Wigner function of the number state $|j\rangle$ is given by

$$W_{|j\rangle}(\beta) = \frac{2}{\pi}\,\exp(-2|\beta|^2)(-1)^j L_j(4|\beta|^2), \qquad (4.15)$$

where $L_j(z)$ stands for the Laguerre polynomial [11] and so the Wigner function of the NCS can be written as

$$W_{\rm NCS}(x, y) = \frac{2}{\pi}\,\exp\big[-2(x^2+y^2)\big]\frac{1}{(1+\bar{n})^N} \times \sum_{m=0}^{\infty}\frac{(N+m)!}{N!\,m!}\,P_m^B(\bar{n})(-1)^{N+m}L_{N+m}\big[4(x^2+y^2)\big] \qquad (4.16)$$

Naturally, when $N = 0$ (4.16) becomes the Wigner function of the chaotic state

$$W_{\rm Ch}(x, y) = \frac{1}{\pi}\,\frac{2}{1+2\bar{n}}\,\exp\left(-\frac{2(x^2+y^2)}{1+2\bar{n}}\right), \qquad (4.17)$$

which has a Gaussian form reducing to that of the vacuum state as $\bar{n} \to 0$.

Both the Q- and Wigner functions of the NCS are symmetric around the origin, as occurs to all mixtures of number states since they have random phase. For $N \neq 0$, the Wigner function varies gradually, as $\bar{n}$ is increased, from the shape of an 'active volcano' (characteristic of a number state) to that of an nonactive one, somehow similar to the form of the Q-function, as illustrated in figure 5. Apparently, for a given $N \neq 0$, the NCS changes its nature from being a quantum state at low values of $\bar{n}$, the Wigner function possessing negative values, to becoming classical for higher $\bar{n}$. However, a detailed analysis of the crater of the volcano show that the Wigner function never becomes non-negative and thus the NCS will not reduce exactly to a nonclassical state as $\bar{n}$ grows. This leads to the question concerning the nonclassical depth of the NCS, which is the point addressed in the next section.

4.3. R representation and nonclassical depth of the NCS

To discuss the nonclassical depth of the NCS, we consider the Lee procedure [30] and introduce the one parameter representation, interpolating between the P- and Q-functions, given by

$$R(\beta, s) = \frac{1}{\pi s}\int {\rm d}^2\eta\; P(\eta)\,{\rm e}^{-|\eta-\beta|^2/s} \qquad (4.18)$$

with $\beta = x + {\rm i}y$, which reduces to the P-, Wigner and Q-functions for the values $s = 0,\ 1/2,\ 1$ respectively. The nonclassical depth of a given state is defined as the minimum ($s_{\rm m}$) among the values of s for which the R-function becomes strictly non-negative. Naturally, $0 \leqslant s_{\rm m} \leqslant 1$ with 0 being the classical value, associated with coherent states for example, and 1 corresponding to the maximum degree of nonclassicality, as for a number state.

For a number state $|j\rangle$, one has [30]

$$R_{|j\rangle}(\beta, s) = \frac{1}{\pi s}\,\exp\left(-|\beta^2|/s\right)\left(-\frac{(1-s)}{s}\right)^j L_j\left(\frac{|\beta|^2}{s(1-s)}\right) \qquad (4.19)$$

and, therefore,

$$R_{\rm NCS}(x, y, s) = \frac{1}{\pi s}\,\exp\big[-\big(x^2+y^2\big)/s\big]\frac{1}{(1+\bar{n})^N} \times \sum_{m=0}^{\infty}\frac{(N+m)!}{N!\,m!}\,P_m^B(\bar{n})\left(-\frac{(1-s)}{s}\right)^{N+m} \times L_{N+m}\left[\frac{(x^2+y^2)}{s(1-s)}\right]. \qquad (4.20)$$

The value of this function at the origin when $1/2 < s < 1$,

$$R_{\rm NCS}(0, 0, s) = \frac{1}{\pi}(-1)^N\,\frac{(1-s)^N}{(s+\bar{n})^{N+1}}, \qquad (4.21)$$

shows that, for N odd, the R-function has a negative value at the origin for s in this range; thus, $s_{\rm m} = 1$ and the NCS with N odd is as nonclassical as possible. When N is even, (4.21) is positive and one has to analyse the minimum value of the R-function. As illustrated in figure 6, where profiles of R-functions along the x-axis are plotted for a specific NCS, the minimum value of R (for N even) tends to zero as s approaches 1 but remains negative for all $s < 1$, implying that the NCS are always as nonclassical as possible within the measure discussed. In this way, one sees explicitly that

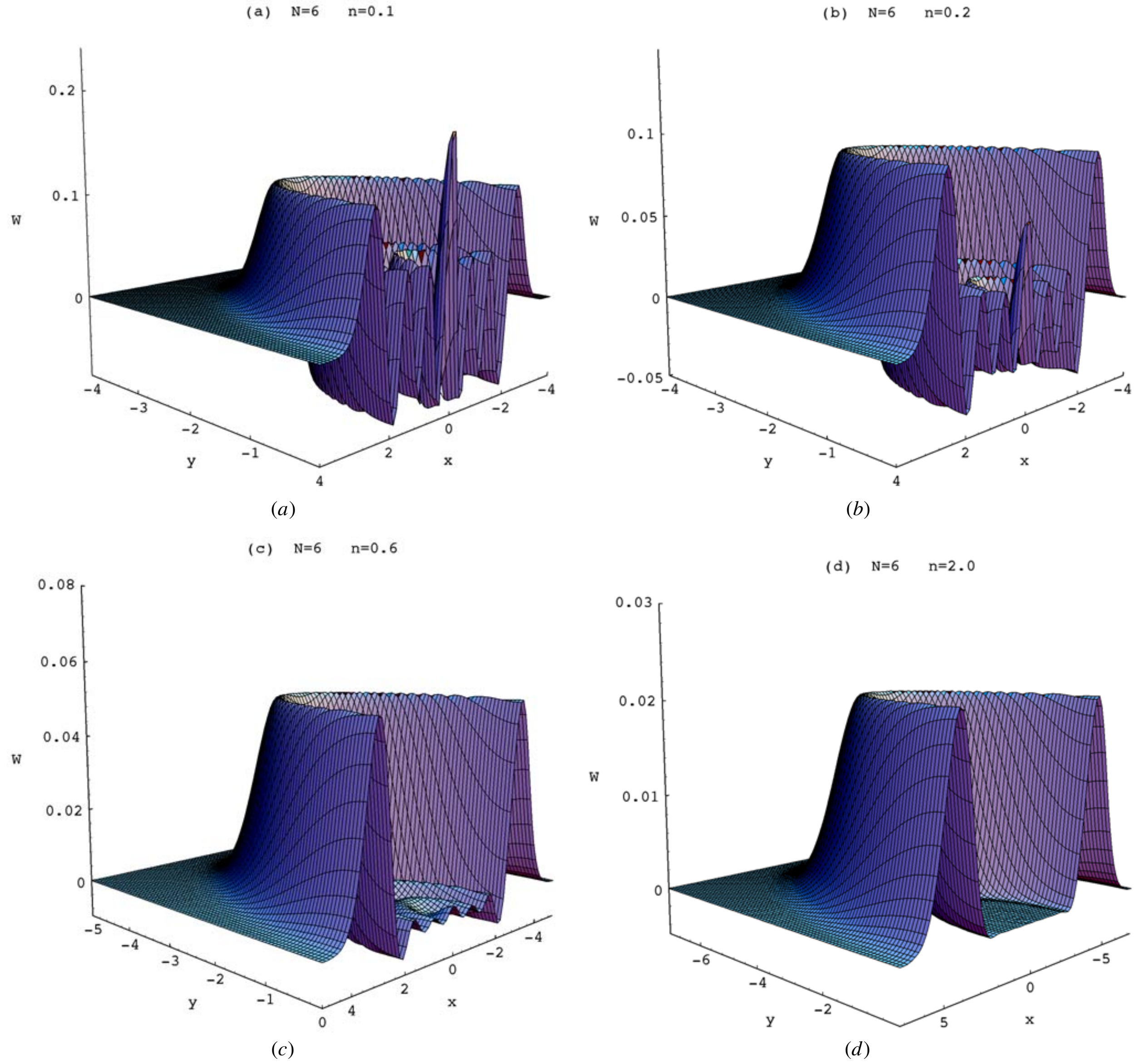


Figure 5. Wigner function of the NCS with $N = 6$ for various values of $\bar{n}$: (*a*) $\bar{n} = 0.1$; (*b*) $\bar{n} = 0.2$; (*c*) $\bar{n} = 0.6$; (*d*) $\bar{n} = 2.0$.

the NCS satisfies Lee's theorem [30] by which a state with a density operator not containing the vacuum component $|0\rangle\langle 0|$ possesses the maximum degree of nonclassicality. Looking at the changes to a given R-function as one increases the chaoticity parameter $\bar{n}$, it is found that its minimum value tends quickly to zero but is negative for all $s < 1$, as occurs with the Wigner function ($s = 1/2$) shown in figure 5; one may say that the quantum fingerprint of the number state remains in the NCS and cannot be completely erased by increasing the degree of chaoticity.

5. Conclusions

We have introduced a new two-parameter interpolating state—designated by NCS—as a truncated mixture of number states of which the density operator $\hat{\rho}_{\mathrm{NCS}}(\bar{n}, N)$ reduces to that of the number state $|N\rangle$ as $\bar{n} \to 0$ and to a chaotic (thermal) state (with temperature specified by $\bar{n}$) when $N = 0$. Such a state is an alternative to the proposal of [16] where interpolation between the number and chaotic states is performed via a pure state—the GGS—which requires a randomization of one of the parameters to obtain the chaotic limit. Here, we already start with a mixed state, the reduction to a pure number state occurring through the elimination of the other constituents of the mixture as $\bar{n}$ goes to zero. Thinking in the opposite direction, one may say that the NCS corresponds to a chaotization of $|N\rangle$, constructed in the same way as the chaotic (thermal) state with the mean number of photons given by $\bar{n}$ (which is given by (2.3), for a thermal state with temperature T), can be built from the vacuum state $|0\rangle$.

The study of the occurrence of nonclassical effects for a field in a NCS reveals that these states do not present quadrature squeezing, which is a feature shared by all mixtures of number states, but they are sub-Poissonian for low values of the chaoticity parameter $\bar{n}$. In fact, we have shown that, as $\bar{n}$ increases from zero to higher values, the Mandel parameter grows from -1 (for $\bar{n} = 0$) tending linearly to infinity as

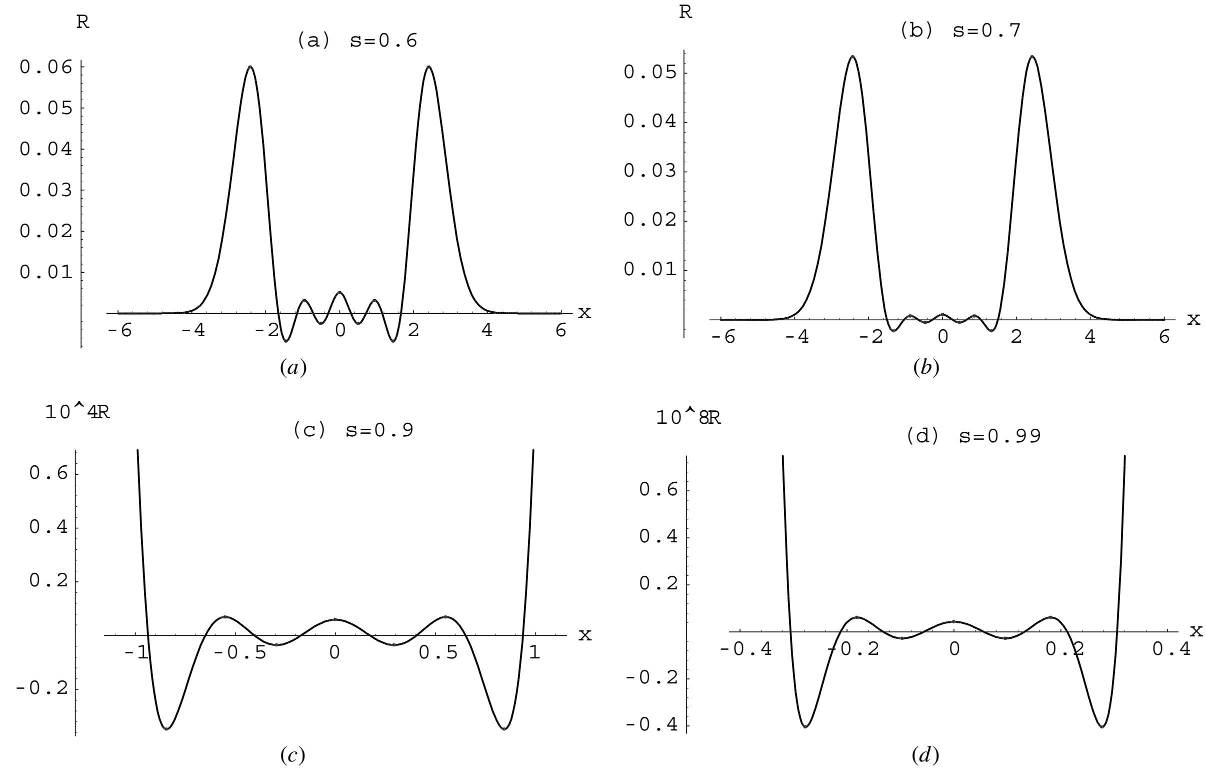


Figure 6. Profiles along the x-axis of some R-functions for the NCS with $N = 4$ and $\bar{n} = 0.5$: (a) $s = 0.6$; (b) $s = 0.7$; (c) $s = 0.9$; (d) $s = 0.99$.

$\bar{n} \to \infty$, a transition between sub- and super-Poissonian characters of the statistics occurring at critical values in the range $1/\sqrt{2} \leqslant \bar{n}_c^{(N)} < 1$. Collapses and revivals of atomic population inversion, for a field initially in a NCS and within the Jaynes–Cummings model, exist and increase with $\bar{n}$ for moderate and large values of N, with a pattern similar to that appearing for a coherent state, but for low N $(= 1, 2)$ the inversion resembles that due to the chaotic (thermal) state.

The analysis of these effects indicates that, for a given N, the NCSs gradually lose their nonclassical properties as the chaoticity parameter $\bar{n}$ is increased. By looking at their phase-space representations, however, one sees that it never becomes a truly classical state. In contrast, it remains as nonclassical as possible in the sense of Lee [30].

A point deserving further investigation in the present perspective would be its extension for the case of interpolation between an arbitrary state and a chaotic state of the radiation field. In the case of a state corresponding to a mixture of Fock states, $\hat{\rho}_{\text{mix}} = \sum_{N=0}^{\infty} p_N |N\rangle\langle N|$, the generalization is straightforward and is similar to the process leading to the squeezed thermal state as a Bose–Einstein weighted sum of the squeezed number states [31]: mixed-chaotic states are obtained replacing the components $|N\rangle\langle N|$ of the NCS by $\hat{\rho}_{\text{NCS}}(\bar{n}, N)$, that is, considering the mixture of NCSs with the same weights. The extension to the case of an arbitrary pure state, however, is more delicate since one has to deal with density matrices possessing non-null off-diagonal elements in the number basis; such an issue is left for future work.

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