Available online at www.sciencedirect.com
ScienceDirect
Physics Letters A 361 (2007) 464–471
PHYSICS LETTERS A
www.elsevier.com/locate/pla

Non-commutative geometry and symplectic field theory

R.G.G. Amorim [a], M.C.B. Fernandes [a], F.C. Khanna [b,c], A.E. Santana [a,*], J.D.M. Vianna [a,d]

[a] Instituto de Física, Universidade de Brasília, 70910-900 Brasília, DF, Brazil
[b] Physics Department, Theoretical Physics Institute, University of Alberta, Edmonton, Alberta T6G 2J1, Canada
[c] TRIUMF, 4004, Westbrook mall, Vancouver, British Columbia V6T 2A3, Canada
[d] Instituto de Física, Universidade Federal da Bahia, Campus de Ondina, 40210-340 Salvador, Bahia, Brazil

Received 7 June 2006; received in revised form 19 July 2006; accepted 27 September 2006

Available online 6 October 2006

Communicated by P.R. Holland

Abstract

In this work we study representations of the Poincaré group defined over symplectic manifolds, deriving the Klein–Gordon and the Dirac equation in phase space. The formalism is associated with relativistic Wigner functions; the Noether theorem is derived in phase space and an interacting field, including a gauge field, approach is discussed.

1. Introduction

Non-commutative geometry has its origin in the Weyl and Moyal works, studying quantization procedures in phase space [1]. Snyder [2] was the first to develop a consistent theory for non-commutative space coordinates, which was based on representations of Lie algebras. Over the last decades there is a revival of non-commutative physics, motivated by some results coming from gravity, condensed matter physics and string theory [3,4]. One particular interest in this context is the development of representation theories for non-commutative fields [5]. However, this type of improvement has not been fully explored in the context of the Weyl–Moyal (phase space) program. In this Letter we address this problem, following a recent work by us [6], where we have considered a representation for the Galilei group on a symplectic manifold. As a result the Schrödinger (not the Liouville–von Neumann) equation was derived in phase space closely associated with the Wigner function. The formalism was used to treat a non-linear oscillator perturbatively and to analyze the concept of coherent states from this phase-space point of view.

The notion of phase space in quantum mechanics arose with the paper by Wigner [7] in order to develop the quantum kinetic theory. In the Wigner formalism, each operator, say A, defined in the Hilbert space, $\mathcal{H}$, is associated with a function, say $a_W(q,p)$, in phase space, Γ. Then there is an application $\Omega_W : A \to a_W(q,p)$, such that, the associative algebra of operators defined in $\mathcal{H}$ turns out to be an associative (but not commutative) algebra in Γ, given by $\Omega_W : AB \to a_W * b_W$, where the star (or Moyal)-product $*$ is defined by [7–10]

$$a_W * b_W = a_W(q,p)\exp\left[\frac{i}{2}\left(\frac{\overleftarrow{\partial}}{\partial q}\frac{\overrightarrow{\partial}}{\partial p} - \frac{\overleftarrow{\partial}}{\partial p}\frac{\overrightarrow{\partial}}{\partial q}\right)\right]b_W(q,p). \tag{1}$$

(Throughout this Letter we use the natural units: $\hbar = c = 1$.) Note that Eq. (1) can be seen as an operator $\hat{A} = a_W *$ acting on functions b_W, such that $\hat{A}(b_W) = a_W * b_W$.

From a mathematical and physical standpoints, the quantum phase space and the Moyal product have been explored along different ways [8–19]. However, it should be of interest to study irreducible unitary representations of kinematical groups considering

* Corresponding author.
E-mail address: asantana@fis.unb.br (A.E. Santana).

0375-9601/$ – see front matter
doi:10.1016/j.physleta.2006.09.074

operators of the type $\hat{A}$. This was our procedure in Ref. [6], and here we extend those representations to the relativistic case with Poincaré–Lie group. In other words, by using the notion of symplectic structure and Moyal product, we construct unitary representations for the Poincaré–Lie algebra, from which we derive the Klein–Gordon and the Dirac equations in phase space. The connection of our formalism with relativistic Wigner functions is then presented, some aspects of interating fields are discussed, including gauge fields, and the Noether theorem is derived in Γ. It is worth mentioning that this alternate version for the field theory provides a way to consider the Wigner approach on the bases of group representation of kinematical symmetries.

The presentation is organized in the following way. In Section 2, we define a Hilbert space $\mathcal{H}(\Gamma)$ over a phase space Γ with its natural relativisitic symplectic structure. $\mathcal{H}(\Gamma)$ will turn out to be the space of representations of kinematical symmetry. In Section 3 we study the Poincaré algebra in $\mathcal{H}(\Gamma)$ and the representations for spin zero and spin 1/2. The Noether theorem is derived in Section 4. The developments of the interacting symplectic field formalism is presented, for a scalar and gauge field, in Section 5; and in Section 6 we present our final concluding remarks.

2. Hilbert space and symplectic structure

Consider M an analytical manifold where each point is specified by Minkowski coordinates q^μ, with $\mu = 0, 1, 2, 3$ and metric specified by $\mathrm{diag}(g) = (-+++)$. The coordinates of each point in the cotangent-bundle T^*M will be denoted by (q^μ, p_μ). The space T^*M is equipped with a symplectic structure via a 2-form

$$\omega = dq^\mu \wedge dp_\mu, \tag{2}$$

called the symplectic form (sum over repeated indices is assumed). We consider the following bidifferential operator on $C^\infty(T^*M)$ functions,

$$\Lambda = \frac{\overleftarrow{\partial}}{\partial q^\mu}\frac{\overrightarrow{\partial}}{\partial p_\mu} - \frac{\overleftarrow{\partial}}{\partial p^\mu}\frac{\overrightarrow{\partial}}{\partial q_\mu}, \tag{3}$$

such that for C^∞ functions, $f(q,p)$ and $g(q,p)$, we have

$$\omega(f\Lambda, g\Lambda) = f\Lambda g = \{f, g\}, \tag{4}$$

where

$$\{f, g\} = \frac{\partial f}{\partial q^\mu}\frac{\partial g}{\partial p_\mu} - \frac{\partial f}{\partial p^\mu}\frac{\partial g}{\partial q_\mu}$$

is the Poisson bracket and $f\Lambda$ and $g\Lambda$ are two vector fields given by $h\Lambda = X_h = -\{h, \}$.

The space T^*M endowed with this symplectic structure is called the phase space, and will be denoted by Γ.

In order to construct a Hilbert space over (this relativistic phase space) Γ, let η be an invariant measure on the cotangent bundle. Then, if φ is a mapping: $\Gamma \to \mathbb{R}$ which is measurable one can define the integral

$$\int_\Omega \varphi(\mathbf{z})\, d\eta(\mathbf{z}) \tag{5}$$

of φ with respect to η, where $\mathbf{z} \in \Gamma$. Let $\mathcal{H}(\Gamma)$ be a linear subspace of the space of *η-measurable* functions $\psi : \Gamma \to \mathbb{C}$ which are square integrable, i.e. such that

$$\int_\Gamma \left|\psi(\mathbf{z})\right|^2 d\eta(\mathbf{z}) < \infty. \tag{6}$$

We can then introduce a Hilbert space inner product, $\langle\cdot|\cdot\rangle$, on $\mathcal{H}(\Gamma)$, as follows:

$$\langle\psi_1|\psi_2\rangle = \int_\Gamma \psi_1(q,p)^\dagger \psi_2(q,p)\, d\eta(q,p), \tag{7}$$

where we take $\mathbf{z} = (q^\mu, p_\mu) = (q, p)$. (We are using the notation $\psi^\dagger(q,p)$ for the complex conjugation.)

Now we take $d\eta(q,p) = d^4p\, d^4q$, such that

$$\int d^4p\, d^4q\, \psi^\dagger(q,p)\psi(q,p) < \infty, \tag{8}$$

which is a real bilinear form. In this case we can write $\psi(q,p) = \langle q, p|\psi\rangle$, with

$$\int d^4p\, d^4q\, |q,p\rangle\langle q,p| = 1, \tag{9}$$

where the kets $|q, p\rangle$ are defined from the set of commuting operators $\bar{Q}$ and $\bar{P}$ defined by [6]

$$\bar{Q}|q, p\rangle = q|q, p\rangle, \qquad \bar{P}|q, p\rangle = p|q, p\rangle.$$

The state of a system will be described by functions $\phi(q, p)$, under the condition

$$\langle\psi|\phi\rangle = \int d^4p\, d^4q\, \psi^\dagger(q, p)\phi(q, p) < \infty. \tag{10}$$

This Hilbert space is denoted by $\mathcal{H}(\Gamma)$.

Let us see how to study the Poincaré group taking $\mathcal{H}(\Gamma)$ as the representation space. For this purpose, we construct unitary transformations $U : \mathcal{H}(\Gamma) \to \mathcal{H}(\Gamma)$ such that $\langle\psi_1|\psi_2\rangle$ is invariant. Starting with Λ defined by Eq. (3), a mapping $e^{i\Lambda/2} = * : \Gamma \times \Gamma \to \Gamma$, called Moyal (or star) product, is defined by

$$f(q, p) * g(q, p) = f(q, p) \exp\left[\frac{i}{2}\left(\overleftarrow{\partial}_q \overrightarrow{\partial}_p - \overleftarrow{\partial}_p \overrightarrow{\partial}_q\right)\right] g(q, p) \tag{11}$$

$$= \exp\left[\frac{i}{2}(\partial_q\partial_{p'} - \partial_p\partial_{q'})\right] f(q, p)g(q', p')|_{q',p'=q,p}, \tag{12}$$

where $f(q, p)$ and $g(q, p)$ are in Γ and $\partial_x = \partial/\partial x$ $(x = p, q)$.

The generators of U can be introduced by the following (Moyal–Weyl) star-operators:

$$\hat{F} = f(q, p)* = f\left(q^\mu + \frac{i}{2}\frac{\partial}{\partial p_\mu}, p^\mu - \frac{i}{2}\frac{\partial}{\partial q_\mu}\right). \tag{13}$$

In the sequence we analyze the Poincaré algebra in this context.

3. Poincaré–Lie algebra in $\mathcal{H}(\Gamma)$

Using the functions q_μ, p_μ, and $m_{\mu\nu} = q_\mu p_\nu - p_\mu q_\nu$ (all of them defined in Γ), and Eq. (13), we construct the corresponding star operators

$$\hat{P}^\mu = p^\mu * = p^\mu - \frac{i}{2}\frac{\partial}{\partial q_\mu}, \tag{14}$$

$$\hat{Q}^\mu = q^\mu * = q^\mu + \frac{i}{2}\frac{\partial}{\partial p_\mu}, \tag{15}$$

and

$$\hat{M}_{\nu\sigma} = M_{\nu\sigma} * = \hat{Q}_\nu \hat{P}_\sigma - \hat{Q}_\sigma \hat{P}_\nu. \tag{16}$$

From this set of unitary operators we obtain, after some long but simple calculations, the following set of commutation relations,

$$[\hat{M}_{\mu\nu}, \hat{P}_\sigma] = i(g_{\nu\sigma}\hat{P}_\mu - g_{\sigma\mu}\hat{P}_\nu),$$
$$[\hat{P}_\mu, \hat{P}_\nu] = 0,$$
$$[\hat{M}_{\mu\nu}, \hat{M}_{\sigma\rho}] = -i(g_{\mu\rho}\hat{M}_{\nu\sigma} - g_{\nu\rho}\hat{M}_{\mu\sigma} + g_{\mu\sigma}\hat{M}_{\rho\nu} - g_{\nu\sigma}\hat{M}_{\rho\mu}).$$

This is the Poincaré algebra, where $\hat{M}_{\mu\nu}$ stand for rotations and $\hat{P}_\mu$ for translations (but notice, in phase space).

The Casimir invariants can be built up from the Pauli–Lubanski matrices, $\hat{W}_\mu = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}\hat{M}^{\nu\sigma}\hat{P}^\rho$, where $\varepsilon_{\mu\nu\rho\sigma}$ is the Levi-Civita symbol. The invariants are:

$$\hat{P}^2 = \hat{P}^\mu \hat{P}_\mu, \tag{17}$$

$$\hat{W} = \hat{W}^\mu \hat{W}_\mu, \tag{18}$$

where $\hat{P}^2$ stands for the mass shell condition and $\hat{W}$ for the spin. In the following we use such a representation to derive equations in phase space for spin 0 and spin 1/2 particles.

To determine the Klein–Gordon field equation, we consider a scalar representation in $\mathcal{H}(\Gamma)$. In this case we can use the invariant $\hat{P}^2$ given in Eq. (17) to write down

$$\hat{P}^2\psi(q, p) = \left(p^2\right) * \phi(q, p) = \left(p^\mu * p_\mu *\right)\phi(q, p) = m^2\phi(q, p),$$

where m is a constant fixing the representation and interpreted as mass, such that the mass shell condition is satisfied. Using Eq. (14) we obtain

$$\left(p^{\mu}p_{\mu} - ip^{\mu}\frac{\partial}{\partial q^{\mu}} - \frac{1}{4}\frac{\partial}{\partial q^{\mu}}\frac{\partial}{\partial q_{\mu}}\right)\phi(q,p) = m^2\phi(q,p), \tag{19}$$

which is the Klein–Gordon equation in phase space. The solution for such equation is

$$\phi(q_\mu, p_\mu) = \xi(p_\mu)e^{-i4p^{\mu}q_{\mu}}, \tag{20}$$

where $\xi(p_\mu)$ is a function that depends on the boundary-conditions.

The Lagrangian that leads to Eq. (19) is given by

$$\pounds = \frac{-1}{4}\frac{\partial\phi}{\partial q_{\mu}}\frac{\partial\phi^{\dagger}}{\partial q^{\mu}} + \frac{1}{2}ip^{\mu}\left(\phi^{\dagger}\frac{\partial\phi}{\partial q^{\mu}} - \phi\frac{\partial\phi^{\dagger}}{\partial q^{\mu}}\right) - \left(p^{\mu}p_{\mu} - m^2\right)\phi\phi^{\dagger}. \tag{21}$$

We use later this Lagrangian to analyze the Noether theorem in phase space and the interaction with gauge fields.

The association of this representation with the Wigner formalism is given by

$$f_W(q,p) = \phi(q,p) * \phi^{\dagger}(q,p),$$

where $f_W(q,p)$ is the relativistic Wigner function. To prove this, we recall that the Klein–Gordon equation in phase space can be written as

$$\hat{P}^2\phi = p^2 * \phi = m^2\phi. \tag{22}$$

Multiplying the right-hand side of the above equation by $\phi^{\dagger}$ we obtain

$$\left(p^2 * \phi\right) * \phi^{\dagger} = m^2\phi * \phi^{\dagger}, \tag{23}$$

but since $\phi^{\dagger} * p^2 = m^2\phi^{\dagger}$, we also have

$$\phi * \left(\phi^{\dagger} * p^2\right) = m^2\phi * \phi^{\dagger}. \tag{24}$$

Subtracting (23) of (24), and using the associativity property of the "$*$" product, we get

$$p^2 * f_W - f_W * p^2 = 0, \tag{25}$$

where the notation $f_W = \phi * \phi^{\dagger}$ has been used.

Eq. (25) can be written as the relativistic Moyal bracket, i.e.

$$\{g, f\}_M = (q,p) * f(q,p) - f(q,p) * g(q,p) = g\left(2\sin\frac{i}{2}\Lambda\right)f.$$

Applying these results to Eq. (25), we obtain

$$p_{\mu}\frac{\partial}{\partial q_{\mu}}f_W = 0, \tag{26}$$

a well known result. Other properties of the Wigner function, such as the non-positiveness, can be derived as in the non-relativistic case; see Ref. [6].

Now let us turn our attention to spin $1/2$ particles. Proceeding as usual, we assume the invariant operator $\gamma^{\mu}\hat{P}_{\mu}$, where $\hat{P}_{\mu}$ is defined by Eq. (14). Thus we write

$$\gamma^{\mu}\hat{P}_{\mu}\psi = \gamma^{\mu}p_{\mu} * \psi = m\psi.$$

Using Eq. (14), we obtain

$$\gamma^{\mu}\left(p_{\mu} - \frac{i}{2}\frac{\partial}{\partial q^{\mu}}\right)\psi = m\psi, \tag{27}$$

which is the Dirac equation in phase space, where the γ^{μ}-matrices fulfill the usual Clifford algebra, $\gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu} = 2g^{\mu\nu}$.

The Lagrangian for such a field can be written as

$$\pounds = \frac{-i}{4}\left(\frac{\partial\bar{\psi}}{\partial q^{\mu}}\gamma^{\mu}\psi - \bar{\psi}\gamma^{\mu}\frac{\partial\psi}{\partial q^{\mu}}\right) - \bar{\psi}\left(m - \gamma^{\mu}p_{\mu}\right)\psi, \tag{28}$$

where $\bar{\psi} = \psi^{\dagger}(q,p)\gamma_0$, with $\psi^{\dagger}(q,p)$ being the Hermitian adjoint of $\psi(q,p)$. The Wigner function in this case is given by $f_W(q,p) = \psi(q,p) * \bar{\psi}(q,p)$, with each component satisfying Eq. (26).

4. Noether theorem in phase space

Our proposal in this section is to demonstrate the Noether theorem in phase space, stating the following,

Noether theorem in $\mathcal{H}(\Gamma)$. *To all the Lie-transformation group changing $\Psi(q,p)$ but leaving $\pounds(\Psi,\partial\Psi)$ invariant up to a divergence, there exists a conserved current in phase space. The divergence is defined by*

$$\left(\frac{\partial}{\partial q^{\mu}}+\frac{\partial}{\partial p^{\mu}}\right)S^{\mu}.$$

Here Ψ stands for the Klein–Gordon field, ϕ, or the Dirac field, ψ.

Proof. Consider the infinitesimal transformation $\Psi(q,p)\to\Psi'(q,p)=\Psi(q,p)+\delta\Psi(q,p)$, such that $\pounds\to\pounds+\delta\pounds$, where $\delta\pounds=(\frac{\partial}{\partial q^{\mu}}+\frac{\partial}{\partial p^{\mu}})S^{\mu}$. Then,

$$\delta\pounds=\frac{\partial\pounds}{\partial\Psi}\delta\Psi+\frac{\partial\pounds}{\partial(\frac{\partial\Psi}{\partial q^{\mu}})}\frac{\partial\delta\Psi}{\partial q^{\mu}}+\frac{\partial\pounds}{\partial(\frac{\partial\Psi}{\partial p^{\mu}})}\frac{\partial\delta\Psi}{\partial p^{\mu}}.$$

Developing this equation and using the Euler–Lagrange equation of motion in phase space, i.e.

$$\frac{\partial\pounds}{\partial\Psi}-\frac{\partial}{\partial q^{\mu}}\frac{\partial\pounds}{\partial(\frac{\partial\Psi}{\partial q^{\mu}})}-\frac{\partial}{\partial p^{\mu}}\frac{\partial\pounds}{\partial(\frac{\partial\Psi}{\partial p^{\mu}})}=0,$$

we get

$$\begin{aligned}\delta\pounds&=\left(\frac{\partial}{\partial q^{\mu}}\frac{\partial\pounds}{\partial(\frac{\partial\Psi}{\partial q^{\mu}})}\delta\Psi+\frac{\partial}{\partial p^{\mu}}\frac{\partial\pounds}{\partial(\frac{\partial\Psi}{\partial p^{\mu}})}\delta\Psi\right)+\left(\frac{\partial\pounds}{\partial(\frac{\partial\Psi}{\partial q^{\mu}})}\frac{\partial\delta\Psi}{\partial q^{\mu}}+\frac{\partial\pounds}{\partial(\frac{\partial\Psi}{\partial p^{\mu}})}\frac{\partial\delta\Psi}{\partial p^{\mu}}\right)\\&=\frac{\partial}{\partial q^{\mu}}\left(\frac{\partial\pounds}{\partial(\frac{\partial\Psi}{\partial q^{\mu}})}\delta\Psi\right)+\frac{\partial}{\partial p^{\mu}}\left(\frac{\partial\pounds}{\partial(\frac{\partial\Psi}{\partial p^{\mu}})}\delta\Psi\right).\end{aligned}$$

As a consequence we obtain

$$\left(\frac{\partial}{\partial q^{\mu}}+\frac{\partial}{\partial p^{\mu}}\right)j^{\mu}=0,$$

where the conserved current is

$$j^{\mu}=\frac{\partial\pounds}{\partial(\frac{\partial\Psi}{\partial q^{\mu}})}\delta\Psi+\frac{\partial\pounds}{\partial(\frac{\partial\Psi}{\partial p^{\mu}})}\delta\Psi-S^{\mu}.$$

This provides a proof of the existence of the Noether-current in phase space. Let us use this result to study some particular transformations. □

Example (i): Γ-time translations. Consider

$$q^{\mu}\to q'^{\mu}=q^{\mu}+\varepsilon^{\mu},\qquad p^{\mu}\to p'^{\mu}=p^{\mu}+\lambda^{\mu}.$$

with the notation $\eta^{\mu}=(q^{\mu},p^{\mu})$, as well as $\epsilon^{\mu}=(\varepsilon^{\mu},\lambda^{\mu})$. Then we obtain

$$\left(\frac{\partial}{\partial q^{\mu}}+\frac{\partial}{\partial p^{\mu}}\right)\theta^{\mu\nu}=0,$$

where the Γ-stress tensor, $\theta^{\mu\nu}$, is given by

$$\theta^{\mu\nu}=\frac{\partial\pounds}{\partial(\frac{\partial\Psi}{\partial\eta^{\mu}})}\frac{\partial\Psi}{\partial\eta^{\nu}}-g^{\mu\nu}\pounds.$$

Taking, as an example, space translations only, that is, $\lambda=0$, we obtain for the free Klein–Gordon field

$$\theta^{\mu\nu}_{\mathrm{KG}}=\frac{-1}{4}\left(\frac{\partial\phi^{*}}{\partial q_{\mu}}\frac{\partial\phi}{\partial q_{\nu}}+\frac{\partial\phi}{\partial q_{\mu}}\frac{\partial\phi^{*}}{\partial q_{\nu}}\right)+\frac{1}{2}ip^{\mu}\left(\phi^{*}\frac{\partial\phi}{\partial q_{\nu}}-\phi\frac{\partial\phi^{*}}{\partial q_{\nu}}\right)-g^{\mu\nu}\pounds.$$

For the Dirac field we obtain

$$\theta^{\mu\nu}_{D}=\frac{-i}{4}\left(-\bar{\psi}\gamma^{\mu}\frac{\partial\psi}{\partial q_{\nu}}+\gamma^{\mu}\psi\frac{\bar{\psi}}{\partial q_{\nu}}\right)-g^{\mu\nu}\pounds.$$

In these two examples, we obtain the usual invariant 4-momentum $P^{\nu}=\int\theta^{0\nu}\,d^{3}q\,d^{4}p$.

Example (ii): Γ-time rotations. We define rotations in Γ by $q^\nu \to q^\nu + \epsilon^{\mu\nu} q_\nu$, and $p^\nu \to p^\nu + \lambda^{\mu\nu} p_\nu$, such that $\delta q^\mu = \epsilon^{\mu\nu} q_\nu$, and $\delta p^\mu = \lambda^{\mu\nu} p_\nu$, with $\epsilon^{\mu\nu} = -\epsilon^{\nu\mu}$, and $\lambda^{\mu\nu} = -\lambda^{\nu\mu}$. Then we find

$$\left(\frac{\partial}{\partial q^\mu} + \frac{\partial}{\partial p^\mu}\right) M^{\mu\nu\lambda} = 0,$$

where $M^{\mu\nu\lambda}$ is a Γ-angular-momentum tensor given by

$$M^{\mu\nu\lambda} = -q^\lambda \theta^{\mu\nu} - p^\lambda \theta^{\mu\nu} + q^\nu \theta^{\mu\lambda} + p^\nu \theta^{\mu\lambda} + i \frac{\partial \pounds}{\partial(\frac{\partial \Psi}{\partial q^\mu})} I^{\nu\lambda} \Psi + i \frac{\partial \pounds}{\partial(\frac{\partial \Psi}{\partial p^\mu})} I^{\nu\lambda} \Psi,$$

with $I^{\mu\nu}$ being such that

$$\delta \Psi = -\frac{\epsilon^{\mu\nu}}{2} (\eta_\nu \partial_\mu - \eta_\mu \partial_\nu - i I_{\mu\nu}) \Psi(\eta).$$

For the Klein–Gordon and Dirac fields, respectively, we thus have

$$M^{\mu\nu\lambda}_{\mathrm{KG}} = -q^\lambda \theta^{\mu\nu}_{\mathrm{KG}} - p^\lambda \theta^{\mu\nu}_{\mathrm{KG}} + q^\nu \theta^{\mu\lambda}_{\mathrm{KG}} + p^\nu \theta^{\mu\lambda}_{\mathrm{KG}} + i \frac{\partial \phi^*}{\partial q_\mu} I^{\nu\lambda} \phi + i \frac{\partial \phi}{\partial q_\mu} I^{\nu\lambda} \phi^* - \frac{1}{2} p^\mu \phi^* I^{\nu\lambda} \phi + \frac{1}{2} p^\mu \phi I^{\nu\lambda} \phi^*,$$

and

$$M^{\mu\nu\lambda}_D = -q^\lambda \theta^{\mu\nu}_D - p^\lambda \theta^{\mu\nu}_D + q^\nu \theta^{\mu\lambda}_D + p^\nu \theta^{\mu\lambda}_D - \frac{1}{4} \bar{\psi} \gamma^\mu I^{\nu\lambda} \psi + \frac{1}{4} \gamma^\mu \psi I^{\nu\lambda} \bar{\psi}.$$

The usual (space rotations only) angular momentum is then given by $M^{\mu\nu} = \int M^{\mu\nu 0} \, d^3 q \, d^4 p$.

Example (iii): non-local gauge symmetry. Using the Noether theorem, let us start with the analysis of gauge symmetries in the context of this symplectic field theory. We consider a global gauge transformation given by

$$\Psi' = e^{-i\alpha} \Psi \quad \text{and} \quad \bar{\Psi}' = e^{i\alpha} \bar{\Psi},$$

where α is a real constant. Up to first order, we have

$$\Psi' = (1 - i\alpha) \Psi \quad \text{and} \quad \delta \Psi = -i\alpha \Psi,$$

we then assume $\delta \pounds = 0$. Then for the Klein–Gordon and Dirac field, respectively, we have

$$j^\mu_{\mathrm{KG}} = \frac{i}{4} \left(\phi \frac{\partial \phi^*}{\partial q_\mu} + \phi^* \frac{\partial \phi}{\partial q_\mu} \right) + p^\mu \phi^* \phi, \tag{29}$$

and

$$j^\mu_D = \bar{\psi} \gamma^\mu \psi. \tag{30}$$

For these two cases, we have $Q = \int j^0 \, d^3 q \, d^4 p$.

5. Elements of interacting fields

In this section we consider the problem of interacting fields in phase space. Let us consider the scalar-field Lagrangian given by

$$\pounds = \frac{-1}{4} \frac{\partial \phi}{\partial q_\mu} \frac{\partial \phi^\dagger}{\partial q^\mu} + \frac{1}{2} i p^\mu \left(\phi^\dagger \frac{\partial \phi}{\partial q^\mu} - \phi \frac{\partial \phi^\dagger}{\partial q^\mu} \right) - \left(p^\mu p_\mu - m^2\right) \phi \phi^\dagger + U\left(\phi \phi^\dagger\right),$$

that leads to the following non-linear Klein–Gordon equation,

$$\frac{-1}{4} \frac{\partial^2 \phi}{\partial q^\mu \partial q_\mu} - i p^\mu \frac{\partial \phi}{\partial q^\mu} + \left(p^\mu p_\mu - m^2\right) \phi + V(\phi) = 0,$$

where $V(\phi) = \frac{\partial U(\phi \phi^\dagger)}{\partial \phi^\dagger}$. In order to solve this equation, we can use the Green's function $G = G(q^\mu, q'^\mu, p^\mu, p'^\mu)$ satisfying the equation

$$\frac{-1}{4} \frac{\partial^2 G}{\partial q^\mu \partial q_\mu} - i p^\mu \frac{\partial G}{\partial q^\mu} + \left(p^\mu p_\mu - m^2\right) G = \delta\left(q^\mu - q'^\mu\right) \delta\left(p^\mu - p'^\mu\right).$$

Then a solution of such equation is given by

$$\phi(q^\mu, p^\mu) = \phi_0(q^\mu, p^\mu) + \int d^4q'^\mu\, d^4p'^\mu\, G(q^\mu, q'^\mu, p^\mu, p'^\mu) V(\phi),$$

where $\phi_0(q^\mu, p^\mu)$ is the solution of the homogeneous Klein–Gordon equation in phase space. The poles of the Green's function can be derived by the following. Taking a Fourier transform, such that $q^\mu \to k^\mu$ and $p^\mu \to \xi^\mu$, we obtain,

$$\frac{1}{4}k^2\tilde{G}(k^\mu, \xi^\mu) - p^\mu k_\mu \tilde{G}(k^\mu, \xi^\mu) + (p^\mu p_\mu - m^2)\tilde{G}(k^\mu, \xi^\mu) = 1,$$

where

$$\tilde{G}(k^\mu, \xi^\mu) = \frac{1}{\frac{1}{4}k^2 - p^\mu k_\mu + p^\mu p_\mu - m^2}.$$

Therefore the Green's function is written as

$$G(q^\mu, q'^\mu, p^\mu, p'^\mu) = \frac{1}{(2\pi)^4}\int d^4k^\mu\, d^4\xi^\mu \frac{e^{-ik^\mu q_\mu - i\xi^\mu p_\mu}}{\frac{1}{4}k^2 - p^\mu k_\mu + p^\mu p_\mu - m^2}.$$

With this result, we construct a generating functional, starting then a study of a quantum field theory in phase space. First, we define the Feynman propagator by

$$(\Box + ip\partial + p^2 + m^2 + i\varepsilon)G_0^F(q, p) = -\delta(q, p),$$

where $ip\partial = ip^\mu/\partial q^\mu$. The generator functional is introduce by

$$\begin{aligned} Z_0 &\simeq \int D\phi\, D\tilde{\phi}\, e^{iS} = \int D\phi \exp\left[i\int dq\, dp(\pounds)\right] \\ &= \int D\phi \exp\left\{-i\int dq\, dp\left[\frac{1}{2}\phi(q, p)(\Box + ip\partial + p^2 + m^2 + i\varepsilon)\phi(q, p) - J(q, p)\phi(q, p)\right]\right\}. \end{aligned}$$

Such a functional can be written as

$$Z_0 \simeq \exp\left\{\frac{i}{2}\int dq\, dp\, dq'\, dp'\left[J(q, p)(\Box + ip\partial + p^2 + m^2 + i\varepsilon)^{-1} J(q', p')\right]\right\}. \tag{31}$$

Then we find

$$G_0^F(\eta - \eta'; \beta) = i\frac{\delta^2 Z[J]}{\delta J(\eta)\delta J(\eta')}\bigg|_{J=0},$$

where again we have used the notation, $\eta = (x, p)$.

Another important aspect of the interacting fields, that can be analyzed in phase space with standard methods, is the gauge symmetry. We start with the scalar field considering a local phase transformation given by

$$\phi'(q, p) = e^{-i\alpha(q,p)}\phi(q, p).$$

Taking an infinitesimal α, we have $\phi \to \phi - i\alpha\phi$, $\delta\phi = -i\alpha\phi$, $\delta\phi^\dagger = i\alpha\phi^\dagger$, such that

$$\delta\left[\left(\frac{\partial}{\partial q^\mu} + \frac{\partial}{\partial p^\mu}\right)\phi\right] = -i\left[\left(\frac{\partial}{\partial q^\mu} + \frac{\partial}{\partial p^\mu}\right)\alpha\right]\phi - i\alpha\left(\frac{\partial}{\partial q^\mu} + \frac{\partial}{\partial p^\mu}\right)\phi,$$

and

$$\delta\left[\left(\frac{\partial}{\partial q^\mu} + \frac{\partial}{\partial p^\mu}\right)\phi^\dagger\right] = i\left[\left(\frac{\partial}{\partial q^\mu} + \frac{\partial}{\partial p^\mu}\right)\alpha\right]\phi^\dagger + i\alpha\left(\frac{\partial}{\partial q^\mu} + \frac{\partial}{\partial p^\mu}\right)\phi^\dagger.$$

Using the Lagrangian given in Eq. (21) for the free scalar field, we obtain

$$\delta\pounds = i\left[\frac{i}{4}\left(\phi\frac{\partial\phi^\dagger}{\partial q_\mu} + \phi^\dagger\frac{\partial\phi}{\partial q_\mu}\right) + p^\mu\phi^\dagger\phi\right]\partial_\mu\alpha = j^\mu\partial_\mu\alpha.$$

Therefore, demanding invariance of the Lagrangian under the gauge transformation, we introduce the $U(1)$ gauge field, such that the full Lagrangian reads

$$\pounds_{\text{total}} = \frac{-1}{4}\frac{\partial\phi}{\partial q_\mu}\frac{\partial\phi^\dagger}{\partial q^\mu} + \frac{1}{2}ip^\mu\left(\phi^\dagger\frac{\partial\phi}{\partial q^\mu} - \phi\frac{\partial\phi^\dagger}{\partial q^\mu}\right) - \left(p^\mu p_\mu - m^2\right)\phi\phi^\dagger - ej^\mu A_\mu + e^2 A_\mu A^\mu \phi^\dagger\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}$$
$$= \left(p_\mu\psi - \frac{i}{2}\frac{\partial\phi}{\partial q^\mu} + ieA_\mu\phi\right)\left(p^\mu\phi^\dagger + \frac{i}{2}\frac{\partial\phi^\dagger}{\partial q_\mu} - ieA^\mu\psi^\dagger\right) + m^2\phi^\dagger\phi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}, \tag{32}$$

where $F_{\mu\nu} = \partial A_\nu/\partial q^\mu - \partial A_\mu/\partial q^\nu$.

A covariant derivative can be defined by

$$D_\mu = \left(p_\mu - \frac{i}{2}\frac{\partial}{\partial q^\mu} + ieA_\mu\right),$$

leading to

$$\pounds = D_\mu\phi D^\mu\phi^\dagger + m^2\phi\phi^\dagger - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}.$$

It is important to notice that the minimum coupling is given by $\hat{P}_\mu \to \hat{P}_\mu + ieA_\mu$, with $\hat{P}_\mu = p_\mu - \frac{i}{2}\frac{\partial}{\partial q^\mu}$, as it is expected. In closing, with the definition of a Feynman propagator, consistent perturbation calculations in phase space can be carried out; this is one aspect to be considered in another place.

6. Concluding remarks

In this Letter we have set forth a field theory based on a relativistic Hilbert phase space, using as a basic ingredient the Moyal product of the non-commutative geometry. We develop a representation theory for kinematical Lie groups as it was stated by us in [6], considering the Galilei group; and in this sense our procedure generalizes the Curtright and Zachos approach [20]. As a consequence we have derived from the Poincaré Lie algebra in phase space: the Klein–Gordon equation (see Eq. (19)), the Dirac equation (see Eq. (27)), the Noether theorem, and the association of these results with the Wigner function formalism.

Furthermore, we extend the formalism to treat interacting fields. First a $\lambda\phi^4(q,p)$ theory is analyzed: we define the proper propagator for use in perturbative methods and introduce a quantization scheme, delineated by the definition of a generating functional. Second, we consider the Abelian, $U(1)$, symmetry, suggesting a generalization to non-Abelian gauge fields. One central point to be emphasized is that the approach developed here permits the calculation of Wigner functions for relativistic systems with methods, based on symmetry, similar to those used in quantum field theory, including prescriptions for diagrammatic analysis. A more detailed account of this will be considered in a longer paper.

Acknowledgements

This work was partially supported by CNPq of Brazil and NSERC of Canada.

References

[1] H. Weyl, Z. Phys. 46 (1927) 1;
J.E. Moyal, Proc. Cambridge Philos. Soc. 45 (1949) 99.
[2] H.S. Snyder, Phys. Rev. 71 (1947) 38.
[3] R.J. Szabo, Phys. Rep. 378 (2003) 207.
[4] N. Seiberg, E. Witten, JHEP 9909 (1999) 032, hep-th/9908142.
[5] R. Amorim, F.A. Farias, Phys. Rev. D 69 (2004) 045013.
[6] M.D. Oliveira, M.C.B. Fernandes, F.C. Khanna, A.E. Santana, J.D.M. Vianna, Ann. Phys. (N.Y.) 312 (2004) 492.
[7] E. Wigner, Phys. Rev. 40 (1932) 749.
[8] M. Hillery, R.F. O'Connell, M.O. Scully, E.P. Wigner, Phys. Rep. 106 (1984) 121.
[9] Y.S. Kim, M.E. Noz, Phase Space Picture and Quantum Mechanics—Group Theoretical Approach, World Scientific, London, 1991.
[10] T. Curtright, D. Fairlie, C. Zachos, Phys. Rev. D 58 (1998) 25002.
[11] D. Galetti, A.F.R. de Toledo Piza, Physica A 214 (1995) 207.
[12] L.P. Horwitz, S. Shashoua, W.C. Schive, Physica A 161 (1989) 300.
[13] P.R. Holland, Found. Phys. 16 (1986) 701.
[14] M.C.B. Fernandes, J.D.M. Vianna, Braz. J. Phys. 28 (1999) 2.
[15] M.C.B. Fernandes, A.E. Santana, J.D.M. Vianna, J. Phys. A: Math. Gen. 36 (2003) 3841.
[16] A.E. Santana, A. Matos Neto, J.D.M. Vianna, F.C. Khanna, Physica A 280 (2001) 405.
[17] D. Bohm, B.J. Hiley, Found. Phys. 11 (1981) 179.
[18] M.C.B. Andrade, A.E. Santana, J.D.M. Vianna, J. Phys. A: Math. Gen. 33 (2000) 4015.
[19] M.A. de Gosson, J. Phys. A: Math. Gen. 38 (2000) 1.
[20] T. Curtright, C. Zachos, J. Phys. A 32 (1999) 771.