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Spinless Duffin-Kemmer-Petiau Oscillator in a Galilean Non-commutative Phase Space

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Abstract We examine Galilei-invariant linear wave equations in a non-commutative phase space. Specifically, we establish and solve the Galilean covariant Duffin-Kemmer-Petiau equation for spin-0 fields in a harmonic oscillator potential. We obtain these wave equations with a Galilean covariant approach, based on a $(4+1)$-dimensional manifold with light-cone coordinates followed by a reduction to a $(3+1)$-dimensional spacetime. We find the exact wave functions and their energy levels, and we examine the effects of non-commutativity.

Keywords Galilean covariance · Non-commutative phase space · Duffin-Kemmer-Petiau equations

1 Introduction

In this paper, we exploit a higher-dimensional formulation of Galilean covariance to study the non-relativistic Duffin-Kemmer-Petiau (DKP) oscillator for a spin-zero field in a non-commutative phase space; that is, where both coordinates and momenta are non commuting. The DKP wave equation, which is of first order, can be seen as a counterpart of the Dirac

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equation for spin-zero and spin-one fields. Its form is similar to the Dirac equation with the gamma matrices replaced by matrices which satisfy the so-called DKP algebra [1–5]. The fact that the DKP equation has not received much attention in the literature might be explained by the equivalence between the Klein-Gordon equation and the DKP equation, and the more complex algebraic structure of the latter [6, 7]. Over the years, that equivalence has been challenged; some of these claims have allegedly been put to rest in Ref. [8]. The relativistic DKP oscillator is discussed, for instance, in Ref. [9, 10].

As far as we know, the first paper on the idea that configuration-space coordinates do not commute was published by Snyder in 1947 [11, 12]. According to Ref. [13–16], the idea first came to Heisenberg in the late 1930s as a possible cure for short-distance singularities. Heisenberg mentioned his idea to Peierls, who relayed it to Pauli, who in turn mentioned it to Oppenheimer, who asked his student H Snyder to develop this idea. The recent interest in non-commutative quantum mechanics was motivated by studies of the low-energy effective theory of D-branes in the background of a Neveu-Schwarz B-field in a non-commutative space [17–20]. Among recent applications, let us mention the quantum Hall effect on non-commutative spaces [21–24], the Landau problem on the non-commutative plane [25–28], planar quantum systems with central potentials [29, 30], and studies of the relativistic DKP oscillator in a non-commutative space [31–35]. Papers investigating Galilei-invariant systems with non-commutative geometry are in Refs. [36–41].

Our main interest in the present problem stems from the connection between non-commutative coordinates and discrete space-time, following the original paper by Snyder [11, 12]. We expect that a Galilean version should be of interest in condensed matter physics for the study of non-relativistic lattice models. Particle physics and condensed matter physics share many tools of quantum field theory, for instance: gauge invariance, spontaneous symmetry breaking, Goldstone bosons, and so on. The Galilean covariance with a metric in an extended manifold is but one further unifying feature. It consists in enforcing Lorentz-like covariance (ubiquitous in high-energy physics) in a $(4+1)$-dimensional manifold in such a way that the resulting theory is Galilean invariant (encountered in condensed matter physics and low-energy physics). Note that in this paper, a $(4+1)$ manifold refers to a $(3,1)$ space-time augmented by 1 space-like coordinate.

A Galilean covariant theory is obtained by the addition of an extra coordinate, s or x^5, embedded in a $(4+1)$ Minkowski manifold [42–44]. This extended manifold consists of five-vectors with coordinates

$$x^\mu = \left(x^1, x^2, x^3, x^4, x^5\right) = (\mathbf{r}, t, s),$$

which transform under Galilean boosts as

$$\begin{aligned}
\mathbf{r}' &= \mathbf{r} - \mathbf{v}t, \\
t' &= t, \\
s' &= s - \mathbf{r}\cdot\mathbf{v} + \frac{1}{2}\mathbf{v}^2 t.
\end{aligned}$$

This transformation leaves invariant the scalar product

$$(\mathbf{r}, t, s)\cdot\left(\mathbf{r}', t', s'\right) \equiv \mathbf{r}\cdot\mathbf{r}' - ts' - t's,$$

defined by the following metric,

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}. \tag{1}$$

Hereafter we shall refer to this as the *Galilean metric*, even though this is equivalent to the Lorentz metric in $(4+1)$ space-time. The term "Galilean" describes the procedure which consists in projecting down to four space-time dimensions, thereby obtaining a Galilean theory. We note that the extra coordinate, s, appears to be related to the quasi-invariance of the free particle Lagrangian under Galilean transformations, since it transforms like the phase of the quantum wavefunction that ensures the invariance of the Schrödinger equation under Galilean transformations [42–44]. If we consider "energy-mass eigenstates" Ψ that satisfy $\mathrm{i}\hbar\partial_4\Psi = E\Psi$ and, in an analogous manner, $\mathrm{i}\hbar\partial_5\Psi = m\Psi$, then we obtain

$$p_\mu = -\mathrm{i}\hbar\partial_\mu = (\mathbf{p}, -E, -m), \tag{2}$$

so that $p^4 = -p_5 = m$ is the mass, and $p^5 = -p_4 = E$ is the energy. Thus, it suggests that x^5 could be seen as being conjugate to m, similarly to time-energy conjugation relation. (The consequences of this interpretation—including a "mass-x^5 uncertainty principle"—remain to be explored.)

The relativistic analogue of the present work is described in Ref. [31], and we shall compare our results with it. Let us consider the usual position and momentum operators, r_i and p_i, which satisfy the canonical commutations relations:

$$[r_i, r_j] = 0, \qquad [p_i, p_j] = 0, \qquad [r_i, p_j] = \mathrm{i}\hbar\delta_{ij}.$$

Following Ref. [31], we consider a non-commutative space described by the operators $\hat{r}_i$ and $\hat{p}_i$:

$$\hat{r}_i = r_i - \frac{\Theta_{ij}}{2\hbar} p_j = r_i + \frac{(\boldsymbol{\Theta}\times\mathbf{p})_i}{2\hbar}, \tag{3}$$

$$\hat{p}_i = p_i + \frac{\Omega_{ij}}{2\hbar} r_j = p_i - \frac{(\boldsymbol{\Omega}\times\mathbf{r})_i}{2\hbar}. \tag{4}$$

They satisfy the following commutation relations:

$$[\hat{r}_i, \hat{r}_j] = \mathrm{i}\Theta_{ij}, \qquad [\hat{p}_i, \hat{p}_j] = \mathrm{i}\Omega_{ij}, \qquad [\hat{r}_i, \hat{p}_j] = \mathrm{i}\hbar\Delta_{ij}, \tag{5}$$

with $\Theta_{ij} = \epsilon_{ijk}\Theta_k$, $\Omega_{ij} = \epsilon_{ijk}\Omega_k$, where Θ_i and Ω_i $(i = 1, 2, 3)$ are real parameters. As mentioned in Ref. [32] (see also Ref. [20, 45]), the bounds on the non-commutativity parameters are currently given by

$$\Theta < 4\times 10^{-40}\ \mathrm{m}^2, \qquad \Omega < 1.76\times 10^{-61}\ \mathrm{kg}^2\,\mathrm{m}^2/\mathrm{s}^2.$$

The matrix Δ_{ij} is given by

$$\Delta_{ij} = \left(1 + \frac{\boldsymbol{\Theta}\cdot\boldsymbol{\Omega}}{4\hbar^2}\right)\delta_{ij} - \frac{\Omega_i\Theta_j}{4\hbar^2}.$$

From the experimental bounds on Θ and Ω, we see that the second term in the parenthesis is less than 10^{-33}.

Our purpose is to apply the $(4+1)$-dimensional Galilean covariant formalism to define the non-relativistic non-commutative DKP oscillator for spinless fields. In Sect. 2, we begin by outlining the commutative version of the Galilean covariant DKP equation. Then we write its non-commutative version and solve it. In both commutative and non-commutative cases, we can use projection operators, developed for the Galilean covariant DKP equation in Ref. [46].

2 Galilean DKP Oscillator in a Commutative Space

We begin this section by reviewing the Galilean DKP formulation in the commutative phase space. In Sect. 2.1, we recall from Refs. [47–49] the spinless field representation. In Sect. 2.2, we apply the projection operators of the Galilean DKP fields and focus on the spin-zero field [46]. We shall establish and discuss solutions of the DKP equations for the *non-commutative* Galilean covariant oscillator in Sect. 3.

The Lagrangian density for the *Galilean covariant* free DKP field Ψ in $(4+1)$ dimensions is given by

$$\mathcal{L} = \frac{1}{2}\overline{\Psi}\beta^{\mu}\partial_{\mu}\Psi - \frac{1}{2}\partial_{\mu}\overline{\Psi}\beta^{\mu}\Psi - k\overline{\Psi}\Psi, \quad \mu = 1, \dots, 5. \tag{6}$$

The adjoint of the spinor field Ψ is denoted $\overline{\Psi}$. It is defined by $\overline{\Psi} = \Psi^{\dagger}\eta$ where

$$\eta = \left(\beta^4 + \beta^5\right)^2 + 1. \tag{7}$$

In Eq. (6), k is a constant, and β^{μ} are matrices that satisfy the DKP algebra [1–5, 50]

$$\beta^{\mu}\beta^{\nu}\beta^{\rho} + \beta^{\rho}\beta^{\nu}\beta^{\mu} = g^{\mu\nu}\beta^{\rho} + g^{\rho\nu}\beta^{\mu},$$

with the metric $g_{\mu\nu}$ given by Eq. (1). The Lagrangian in Eq. (6) leads to the Galilean DKP wave equation and its adjoint:

$$\begin{aligned} \left(\beta^{\mu}\partial_{\mu} + k\right)\Psi &= 0, \\ \overline{\Psi}\left(\beta^{\mu}\overleftarrow{\partial}_{\mu} - k\right) &= 0. \end{aligned} \tag{8}$$

With appropriate representations of the β-matrices, these equations describe spinless and spin-one fields (see detail in Refs. [47–49]). The β-matrices are given by representations of the Lie algebra so(5,1); this is analogous to the representations of so(4,1) in a 4-dimensional space-time. For the Galilean DKP wave equations, the relevant representations are six-dimensional for spinless fields (in Sect. 2.1), and 15-dimensional for spin-one fields. We will examine the spin-one field with an oscillator in a separate paper.

2.1 DKP-Oscillator Wave Equation

In Ref. [49], we utilized the following 6-by-6 representation for the spin-zero DKP field:

$$\begin{aligned} \beta^1 &= e_{1,6} + e_{6,1}, \\ \beta^2 &= e_{2,6} + e_{6,2}, \\ \beta^3 &= e_{3,6} + e_{6,3}, \\ \beta^4 &= e_{4,6} - e_{6,5}, \\ \beta^5 &= e_{5,6} - e_{6,4}. \end{aligned}$$

The notation e_{jk} is a shorthand for square matrices whose only non-zero entry is jk; that is, $(e_{jk})_{mn} \equiv \delta_{jm}\delta_{kn}$.

The spin-zero oscillator in described by substituting these matrices into Eq. (8), acting of the 6-vector $\Psi = (\psi_1, \dots, \psi_6)^t$, where t denotes transpose. The momentum representation of Eq. (8) is

$$\left(\beta^{\mu}p_{\mu} - \mathrm{i}k\right)\Psi = 0,$$

into which we insert the non-minimal coupling,

$$\mathbf{p} \to \mathbf{p} + \mathrm{i}m\omega\eta\mathbf{r}. \tag{9}$$

The explicit form becomes

$$\left[\boldsymbol{\beta} \cdot (\mathbf{p} + \mathrm{i}m\omega\eta\mathbf{r}) + \beta^4 p_4 + \beta^5 p_5 - \mathrm{i}k\right]\Psi = 0,$$

which leads to the equations

$$\begin{aligned} -\mathrm{i}k\psi_j + (\mathbf{p}_j - \mathrm{i}m\omega\mathbf{r}_j)\psi_6 &= 0, \quad j = 1, 2, 3, \\ -\mathrm{i}k\psi_4 + p_4\psi_6 &= 0, \\ -\mathrm{i}k\psi_5 + p_5\psi_6 &= 0. \end{aligned} \tag{10}$$

If we proceed as in Refs. [47–49]), we obtain

$$E\psi_6 = \left(\frac{\mathbf{p}^2}{2m} + \frac{1}{2}m\omega^2\mathbf{r}^2 + \frac{3}{2}\hbar\omega\right)\psi_6. \tag{11}$$

This result was obtained in Ref. [49] with the 5-dimensional Galilean covariant formalism, and through a low-velocity limit process from the relativistic DKP equation, in Ref. [51].

2.2 DKP Projectors

Given a general representation of the DKP matrices β^μ, the selection of the scalar or vector sector can be done through projection operators [46]. The spinless sector can be selected by the operator P:

$$P = -\frac{1}{2}\left(\beta^4 + \beta^5\right)^2\left(\beta^1\right)^2\left(\beta^2\right)^2\left(\beta^3\right)^2,$$

which satisfies the properties

$$\begin{aligned} &P^2 = P, \\ &P^\mu = P\beta^\mu, \\ &P^\mu\beta^\nu = Pg^{\mu\nu}, \qquad P^i\eta = P^i, \qquad P\eta = -P. \end{aligned} \tag{12}$$

This operator allows us to write Eq. (8) as

$$\left(\beta^\mu\partial_\mu + k\right)(P\Psi) = 0,$$

where $P\Psi$ transforms like a scalar under Galilean boosts. Note that $P^\mu\Psi$ transforms like a pseudo-vector [46].

Instead of Eq. (9), we can consider general non-minimal couplings, that allow us to describe interactions between scalar bosons and a external vector potential $\mathbf{C}(r)$:

$$\mathbf{p} \to \mathbf{p} + \mathbf{C}\eta.$$

From this coupling, if we consider the action of the operator P on the DKP equation as in Eq. (8), and p_μ as in Eq. (2) and Refs. [47–49], we obtain the wave equation

$$EP\Psi = \frac{1}{2m}\left(\mathbf{p}^2 - \mathbf{C}^2 - \mathrm{i}\nabla \cdot \mathbf{C}\right)P\Psi.$$

Clearly, the oscillator described in Sect. 2.1 corresponds to the special case

$$\mathbf{C} = \mathrm{i}m\omega\mathbf{r}. \tag{13}$$

This leads to the following equation [46]:

$$E(P\Psi) = \left(\frac{\mathbf{p}^2}{2m} + \frac{1}{2}m\omega^2\mathbf{r}^2 + \frac{3}{2}\hbar\omega\right)(P\Psi),$$

in agreement with Eq. (11).

In Sect. 3.3, we shall need the counterpart of Eq. (12),

$${}^{\mu}P = \beta^{\mu}P, \tag{14}$$

such that the wave equations for Ψ and $\overline{\Psi}$ lead to

$$P^{\mu}\Psi = -\frac{1}{\hbar k}\partial^{\mu}P\Psi \tag{15}$$

and

$$\overline{\Psi}^{\mu}P = \frac{1}{\hbar k}\partial^{\mu}\overline{\Psi}P. \tag{16}$$

We shall use these relations, as well as

$$\beta^{\mu} = {}^{\mu}P + P^{\mu}, \tag{17}$$

when we normalize the DKP wave functions.

3 DKP Oscillator in a Non-commutative Space

In this section, we turn to the DKP wave equation in a *non-commutative* phase space. We formulate these equation by substituting into the DKP equation (8) the non-commutative coordinates and momenta, $\hat{r}_i$ and $\hat{p}_i$, given by Eqs. (3) and (4). In Sect. 3.1, we consider a general DKP wave equation and utilize the projector approach to obtain the spin-zero equation. We determine the energy spectrum in Sect. 3.2 via the separation of variables, and describe the normalized wave functions in Sect. 3.3.

3.1 DKP Wave Equation in a Non-commutative Space

The DKP equation with a non-minimal coupling $\mathbf{C}$, in a non-commutative space, is written as

$$\left(\beta^{\mu}\boldsymbol{\pi}_{\mu} - \mathrm{i}\hbar k\right)\Psi = 0, \tag{18}$$

where $\pi_{\mu} = (\hat{\mathbf{p}} + \mathbf{C}\eta, p_4, p_5)$ with $\mathbf{C} = \mathbf{C}(\hat{r})$. If we apply the operators P and P^{μ} to each term in Eq. (18), we obtain

$$\begin{aligned}
\mathrm{i}\hbar k P^{j}\Psi &= \left(\hat{p}^{j} - C^{j}\right)P\Psi,\\
\mathrm{i}\hbar k P^{4}\Psi &= -mP\Psi,\\
\mathrm{i}\hbar k P^{5}\Psi &= -EP\Psi,\\
\mathrm{i}\hbar k P\Psi &= \left((\hat{p}_i + C_i)P^{i} + EP^{4} + mP^{5}\right)\Psi,
\end{aligned}$$

so that Eq. (18) becomes

$$EP\Psi = \frac{1}{2m}\left(\hat{\mathbf{p}}^2 - \mathbf{C}^2 + [\hat{p}_i, C_i]\right)P\Psi. \tag{19}$$

This is the wave equation for the scalar field $P\Psi$ in a non-commutative space with a general non-minimal coupling. In other words, if we have the functional dependence for the vector potential $\mathbf{C}(\hat{r})$ in a non-commutative space, then it is possible to write down the complete wave equation that describes the interaction.

For instance, the free field corresponds to $\mathbf{C}=0$. Then we can recast Eq. (19) as

$$EP\Psi=\frac{1}{2m}\left(\mathbf{p}^2-\frac{1}{\hbar}\boldsymbol{\Omega}\cdot\mathbf{L}+\frac{1}{4\hbar^2}(\mathbf{r}\times\boldsymbol{\Omega})^2+\hbar^2k^2\right)P\Psi.$$

This equation can be interpreted as a non-relativistic free particle in a commutative space with spin-orbit coupling in the presence of a constant magnetic field, given in terms of the non-commutative parameter vector $\boldsymbol{\Omega}$.

Now let us couple the scalar field to the three-dimensional harmonic oscillator in a non-commutative space. From Eq. (19) with the potential given in Eq. (13), we find that Eq. (19) reduces to

$$\begin{aligned}EP\Psi=\frac{1}{2m}\bigg[&\mathbf{p}^2+m^2\omega^2\mathbf{r}^2-3m\hbar\omega-\frac{1}{\hbar}\big(\boldsymbol{\Omega}+m^2\omega^2\boldsymbol{\Theta}\big)\cdot\mathbf{L}\\&+\frac{1}{4\hbar^2}\big((\mathbf{r}\times\boldsymbol{\Omega})^2+m^2\omega^2(\mathbf{p}\times\boldsymbol{\Theta})^2\big)-\frac{m\omega}{2\hbar}\boldsymbol{\Theta}\cdot\boldsymbol{\Omega}+\hbar^2k^2\bigg]P\Psi.\end{aligned}\tag{20}$$

Let us denote the field simply by $\psi\equiv P\Psi$. From now on, we choose the non-commutativity vectors to point in the z-direction,

$$\boldsymbol{\Theta}=(0,0,\Theta),\qquad \boldsymbol{\Omega}=(0,0,\Omega).$$

3.2 Energy Spectrum

Hereafter, we substitute the previous expressions into the explicit representation utilized to obtain Eq. (10), and reduce these equations into a single equation for ψ_6. Equivalently, we can use Eq. (20) and substitute the values of Θ and Ω. With cylindrical coordinates (ρ,ϕ,z), we obtain

$$\begin{aligned}E\psi=&\left[-\left(\frac{\hbar^2}{2m}+\frac{m\omega^2\Theta^2}{8\hbar^2}\right)\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}\right)+\left(\frac{1}{2}m\omega^2+\frac{\Omega^2}{8m\hbar^2}\right)\rho^2\right]\psi\\&+\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial z^2}+\frac{1}{2}m\omega^2z^2-\frac{3}{2}\hbar\omega\right]\psi\\&-\left[\frac{1}{2m\hbar}\big(\Omega+m^2\omega^2\Theta\big)L_3+\frac{\omega}{4\hbar}\Theta\Omega-\frac{\hbar^2k^2}{2m}\right]\psi.\end{aligned}$$

We perform the separation of variables as follows:

$$\psi(\rho,\phi,z)=\chi(\rho)\Phi(\phi)\Xi(z).\tag{21}$$

The function $\Phi(\phi)$ is given by

$$\Phi(\phi)=\exp(\mathrm{i}|m_l|\phi),\tag{22}$$

with m_l given by

$$L_3\psi=m_l\hbar\psi.$$

After dividing each term of Eq. (21) by $\chi(\rho)\Phi(\phi)\Xi(z)$, it becomes

$$\begin{aligned} E = & -\frac{\hbar^2}{2m}\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{d\chi}{d\rho}\right)\frac{1}{\chi} + \left(\frac{\hbar^2}{2m} + \frac{m\omega^2\Theta^2}{8}\right)\frac{m_l^2}{\rho^2} \\ & + \left(\frac{1}{2}m\omega^2 + \frac{\Omega^2}{8m\hbar^2}\right)\rho^2 - \frac{m\omega^2\Theta^2}{8}\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{d\chi}{d\rho}\right)\frac{1}{\chi} \\ & - \frac{\hbar^2}{2m}\frac{d^2\Xi}{dz^2}\frac{1}{\Xi} + \frac{1}{2}m\omega^2 z^2 \\ & - \frac{3}{2}\hbar\omega - \frac{m_l}{2m}\left(\Omega + m^2\omega^2\Theta\right) - \frac{\omega}{4\hbar}\Theta\Omega + \frac{\hbar^2 k^2}{2m}. \end{aligned} \tag{23}$$

Note that the terms of the first two lines on the right-hand side of Eq. (23) depend on ρ only; we set their sum equal to the constant E_ρ. The third line depends on z only; we set it equal to the constant E_{n_z}. The remaining terms (E from the left-hand side, and the fourth line of Eq. (23)) are independent of the coordinates. Thus each set of terms is equal to a constant, and when we separate the variables, the third line of Eq. (23) gives

$$\frac{\hbar^2}{2m}\frac{d^2\Xi}{dz^2} + \left(E_{n_z} - \frac{1}{2}m\omega^2 z^2\right)\Xi(z) = 0, \tag{24}$$

and the first two lines of Eq. (23) lead to

$$\begin{aligned} & \left(\frac{\hbar^2}{2m} + \frac{m\omega^2\Theta^2}{8}\right)\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{d\chi}{d\rho}\right) + \left(E_\rho - \left(\frac{\hbar^2}{2m} + \frac{m\omega^2\Theta^2}{8}\right)\frac{m_l^2}{\rho^2}\right. \\ & \left. - \left(\frac{1}{2}m\omega^2 + \frac{\Omega^2}{8m\hbar^2}\right)\rho^2\right)\chi(\rho) = 0. \end{aligned} \tag{25}$$

The constants E_{n_z} and E_ρ are related to the fourth line of Eq. (23) as follows:

$$E_{n_z} + E_\rho = E + \frac{3}{2}\hbar\omega + \frac{m_l}{2m}\left(\Omega + m^2\omega^2\Theta\right) + \frac{\omega}{4\hbar}\Theta\Omega - \frac{\hbar^2 k^2}{2m}. \tag{26}$$

Of course, Eq. (24) is the one-dimensional Schrödinger equation for the simple harmonic oscillator, whose solution is (for instance, see Chap. 5 of Ref. [52])

$$\Xi(z) = 2^{-n_z/2}(n_z!)^{-1/2}\left(\frac{m\omega}{\hbar\pi}\right)^{1/4}\exp\left(-\frac{m\omega}{2\hbar}z^2\right)H_{n_z}\left(\sqrt{\frac{m\omega}{\hbar}}z\right), \tag{27}$$

where H_{n_z} denotes the Hermite polynomial of degree n_z, with the corresponding energy eigenvalue given by

$$E_{n_z} = \left(n_z + \frac{1}{2}\right)\hbar\omega. \tag{28}$$

Let us return to the radial, or ρ-dependent, part of Eq. (25) by first rewriting it as

$$\left[\frac{\hbar^2}{2M}\left(\frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho} - \frac{m_l^2}{\rho^2}\right) + E_\rho - \frac{1}{2}M\overline{\omega}_{\Theta,\Omega}^2\rho^2\right]\chi(\rho) = 0, \tag{29}$$

where

$$\begin{aligned} M &= \frac{4m\hbar^2}{4\hbar^2 + m^2\omega^2\Theta^2}, \\ \overline{\omega}_{\Theta,\Omega} &= \frac{1}{4m\hbar^2}\sqrt{(4m^2\hbar^2\omega^2 + \Omega^2)(4\hbar^2 + m^2\omega^2\Theta^2)}. \end{aligned} \tag{30}$$

We notice that M becomes equal to m as the non-commutativity parameter Θ approaches zero.

If we change the variable from ρ to

$$y = \frac{M\overline{\omega}_{\Theta,\Omega}}{2\hbar}\rho^2, \tag{31}$$

then Eq. (29) can be cast into the form

$$\left(y\frac{d^2}{dy^2} + \frac{d}{dy} - \frac{m_l^2}{4y} - y + \beta\right)\chi(y) = 0, \tag{32}$$

where

$$\beta = \frac{E_\rho}{\hbar\overline{\omega}_{\Theta,\Omega}}.$$

This equation is the same as in the relativistic DKP equation (see Eq. (22) in Ref. [31]).

Let us introduce the function $\varphi(y)$, given by

$$\chi(y) = e^{-y}y^{|m_l|/2}\varphi(y). \tag{33}$$

If we substitute this into Eq. (32), we obtain the following differential equation for $\varphi(y)$:

$$\left[y\frac{d^2}{dy^2} + (\gamma - 2y)\frac{d}{dy} + \beta - \gamma\right]\varphi(y) = 0,$$

where $\gamma \equiv |m_l| + 1$. By taking $w \equiv 2y$ and $-2\alpha \equiv \beta - \gamma$, we finally obtain

$$w\frac{d^2\varphi}{dw^2} + (\gamma - w)\frac{d\varphi}{dw} - \alpha\varphi = 0.$$

This is Kummer's differential equation, whose solution is given by the confluent hypergeometric function (see Sect. 13.1.1 in Ref. [53]), so that

$$\varphi(w) = N\big[\,{}_1F_1(\alpha;\gamma;w)\big], \tag{34}$$

where N is a normalization constant, and

$${}_1F_1(\alpha;\gamma;w) = 1 + \frac{\alpha w}{\gamma} + \frac{(\alpha)_2 w^2}{(\gamma)_2 2!} + \cdots + \frac{(\alpha)_n w^n}{(\gamma)_n n!} + \cdots,$$

with the Pocchammer symbol defined as

$$(a)_n \equiv a(a+1)(a+2)\cdots(a+n-1), \quad (a)_0 \equiv 1. \tag{35}$$

From the boundary condition, $w \to \infty$ (which follows from $\rho \to \infty$), which implies $\varphi(w) \to 0$ (so that $\psi \to 0$), we obtain

$$\alpha = \frac{1}{2}\left(|m_l| + 1 - \frac{E_\rho}{\hbar\overline{\omega}_{\Theta,\Omega}}\right) = -n_\rho, \quad n_\rho = 0, 1, 2, \ldots$$

so that

$$E_\rho = (2n_\rho + |m_l| + 1)\hbar\overline{\omega}_{\Theta,\Omega}. \tag{36}$$

To summarize, the energy eigenvalue, $E_{n_\rho m_l n_z}$, of the DKP oscillator is obtained by substituting Eqs. (28) and (36) into Eq. (26) and solving for E. If we absorb k within the energy, we find that

$$E_{n_\rho m_l n_z} = (n_z - 1)\hbar\omega + (2n_\rho + |m_l| + 1)\hbar\overline{\omega}_{\Theta,\Omega} - \frac{m_l}{2m}\left(\Omega + m^2\omega^2\Theta\right) - \frac{\omega}{4\hbar}\Theta\Omega, \tag{37}$$

where $\overline{\omega}_{\Theta,\Omega}$ is given in Eq. (30). The resulting energy spectrum is non-degenerate.

3.3 Normalized Wave Functions

The total wave function $\psi(\rho,\phi,z)$, given by Eq. (21) (with $\chi(\rho)$ obtained in Eqs. (33), (31) and (34), $\Phi(\phi)$ given in Eq. (22), and $\Xi(z)$ obtained in Eq. (27)), can be expressed as follows:

$$\psi(\rho,\phi,z) = \overline{N}\,\rho^{|m_l|} e^{\mathrm{i}|m_l|\phi} e^{-\frac{m\omega}{2\hbar}z^2 - \frac{M\overline{\omega}_{\Theta,\Omega}}{2\hbar}\rho^2}\,{}_1F_1\left(-n_\rho; |m_l|+1; \frac{M\overline{\omega}_{\Theta,\Omega}}{\hbar}\rho^2\right) H_{n_z}\left(\sqrt{\frac{m\omega}{\hbar}}z\right),$$

where $\overline{N}$ is given by

$$\overline{N} = N\,2^{-n_z/2}(n_z!)^{-1/2}\left(\frac{m\omega}{\hbar\pi}\right)^{1/4}\left(\frac{M\overline{\omega}_{\Theta,\Omega}}{2\hbar}\right)^{|m_l|/2}.$$

Our normalization follows from the fourth component, j^4, of the conserved current $j^\mu = \frac{\mathrm{i}\hbar k}{2m}\overline{\Psi}\beta^\mu\Psi$, so that we have

$$\frac{\mathrm{i}\hbar k}{2m}\int_0^\infty \overline{\Psi}\beta^4\Psi\rho d\rho d\phi = 1.$$

If we use $\beta^4 = {}^4P + P^4$ from Eq. (17), the previous equation becomes

$$\frac{\mathrm{i}\hbar k}{2m}\int_0^\infty \overline{\Psi}\left({}^4P + P^4\right)\Psi\rho d\rho d\phi = 1,$$

so that when we substitute Eqs. (15) and (16), as well as Eq. (2), in the previous equation, we obtain

$$\frac{\mathrm{i}\hbar k}{2m}\int_0^\infty \overline{\Psi}\left(\frac{\mathrm{i}m}{\hbar k} + \frac{\mathrm{i}m}{\hbar k}\right)P\Psi\rho d\rho d\phi = -\int_0^\infty \overline{\Psi}P\Psi\rho d\rho d\phi = \int_0^\infty \psi^\dagger\psi\rho d\rho d\phi = 1.$$

Note that the Hermite function, which describes the oscillating motion in z, is already properly normalized. Likewise, the exponential in ϕ is already normalized. After integrating over ϕ and ρ, we find

$$(2\pi)2^{-|m_l|}N^2\int_0^\infty \left(\frac{M\overline{\omega}_{\Theta,\Omega}}{\hbar}\rho^2\right)^{|m_l|} e^{-\frac{M\overline{\omega}_{\Theta,\Omega}}{\hbar}\rho^2}\left({}_1F_1\left[a; b; \frac{M\overline{\omega}_{\Theta,\Omega}}{\hbar}\rho^2\right]\right)^2 \rho d\rho = 1.$$

(The factor 2π follows from the integration over ϕ.)

Let us define $x = \frac{M\overline{\omega}_{\Theta,\Omega}}{\hbar}\rho^2$, so that $\rho d\rho = \frac{\hbar}{M\overline{\omega}_{\Theta,\Omega}}dx$. Then we find

$$\frac{N^2\hbar}{2^{|m_l|}M\overline{\omega}_{\Theta,\Omega}}\sum_{i,j=0}^\infty \frac{(a)_i(a)_j}{(b)_i(b)_j i!j!}\int_0^\infty x^{|m_l|+i+j}e^{-x}dx = 1,$$

where the sums are from the Kummer functions and $(a)_n$ is given in Eq. (35). Next, we utilize the integral $\int_0^\infty y^{\alpha-1}e^{-y}dy = \Gamma(\alpha)$, we have

$$\frac{N^2\hbar}{2^{|m_l|}M\overline{\omega}_{\Theta,\Omega}}\sum_{i,j=0}^\infty \frac{(a)_i(a)_j}{(b)_i(b)_j i!j!}\Gamma(|m_l|+i+j+1) = 1.$$

This result can be written in the form

$$\frac{N^2\hbar\Gamma(|m_l|+1)}{2^{|m_l|}M\overline{\omega}_{\Theta,\Omega}}\sum_{i,j=0}^\infty \frac{(|m_l|+1)_{i+j}(a)_i(a)_j}{(b)_i(b)_j i!j!} = 1, \tag{38}$$

as well as

$$\frac{N^2\hbar\Gamma(|m_l|+1)}{2^{|m_l|}M\overline{\omega}_{\Theta,\Omega}}F_2[|m_l|+1,a,a;b,b;1,1]=1, \tag{39}$$

where we have used the following expression for the Appell hypergeometric series:

$$F_2\big[a,b,b';c,c';x,y\big]=\sum_{n,m=0}^{\infty}\frac{(a)_{m+n}(b)_m(b')_n}{(c)_m(c')_n}\frac{x^m}{m!}\frac{x^n}{n!}.$$

On the other hand, the result in Eq. (38) can be rewritten in another way by redefining the index as $i+j=n$; this leads to

$$\begin{aligned}\frac{N^2\hbar}{2^{|m_l|}M\overline{\omega}_{\Theta,\Omega}}\sum_{n=0}^{\infty}\sum_{i=0}^{n}\frac{(a)_i(a)_{n-i}}{(b)_i(b)_{n-i}i!(n-i)!}(|m_l|+n)!=1,\\ \frac{N^2\hbar}{2^{|m_l|}M\overline{\omega}_{\Theta,\Omega}}\sum_{n=0}^{\infty}\sum_{i=0}^{n}\frac{(|m_l|+n)!(a)_i(a)_{n-i}}{(b)_i(b)_{n-i}i!(n-i)!}=1,\end{aligned} \tag{40}$$

which agrees with the coefficient obtained by Yang et al. [31]. Then, we can express the constant N in two forms: first, with Eq. (39),

$$N^2=\frac{2^{|m_l|}M\overline{\omega}_{\Theta,\Omega}}{\hbar\Gamma(|m_l|+1)F_2[|m_l|+1,a,a;b,b;1,1]},$$

or by using Eq. (40),

$$N^2=\frac{2^{|m_l|}M\overline{\omega}_{\Theta,\Omega}}{\hbar}\frac{1}{\sum_{n=0}^{\infty}\sum_{i=0}^{n}\frac{(|m_l|+n)!(a)_i(a)_{n-i}}{(b)_i(b)_{n-i}i!(n-i)!}}.$$

Then $\overline{N}$ is given by

$$\overline{N}=\sqrt{\frac{\frac{1}{\sqrt{\pi^3}}\frac{1}{2^{n_z/2+1}n_z!}(\frac{M\overline{\omega}_{\Theta,\Omega}}{\hbar})^{|m_l|+1}(\frac{m\omega}{\hbar\pi})^{1/2}}{\sum_{n=0}^{\infty}\sum_{i=0}^{n}\frac{(|m_l|+n)!(a)_i(a)_{n-i}}{(b)_i(b)_{n-i}i!(n-i)!}}}.$$

Now let us return to the complete spinor Ψ, given by Eq. (18),

$$\Psi=\frac{1}{\mathrm{i}\hbar k}\beta^{\mu}\pi_{\mu}\Psi.$$

With the expressions (14) and (17), this spinor can be written as

$$\Psi=\frac{1}{\mathrm{i}\hbar k}\big({}^{\mu}P+P^{\mu}\big)\pi_{\mu}\Psi,$$

as well as

$$\Psi=\frac{1}{\mathrm{i}\hbar k}\big[{}^{i}P(\hat{p}_i-C_i)+P^{i}(\hat{p}_i+C_i)+\big({}^{4}P+P^{4}\big)p_4+\big({}^{5}P+P^{5}\big)p_5\big]P\Psi,$$

where the operator $\hat{p}_i$ and C_i are written in terms of cylindrical coordinates. This expression shows us that all we need is to obtain the wave function $P\Psi$, so that all the other components of Ψ are obtained by the derivatives with respect to the coordinates. Also, if we use the 6×6 representation presented at the beginning of Sect. 2.1, we can express the spinor as follows,

$$\Psi = \frac{1}{\mathrm{i}\hbar k}\begin{pmatrix} \hat{p}_1 - C_1 \\ \hat{p}_2 - C_2 \\ p_3 - C_3 \\ p_4 \\ p_5 \\ 1 \end{pmatrix} P\Psi.$$

Next, if we apply

$$\begin{aligned}\hat{p}_1 - C_1 &= -\mathrm{i}\hbar\partial_x + \frac{\Omega y}{2\hbar} - \mathrm{i}m\omega\left(x + \mathrm{i}\hbar\frac{\Theta\partial_y}{2\hbar}\right)\\ &= -\mathrm{i}\hbar\left(\cos\phi\,\partial_\rho - \frac{\sin\phi}{\rho}\partial_\phi\right) + \frac{\Omega}{2\hbar}\rho\sin\phi\\ &\quad - \mathrm{i}m\omega\left(\rho\cos\phi + \mathrm{i}\hbar\frac{\Theta}{2\hbar}\left(\sin\phi\,\partial_\rho + \frac{\cos\phi}{\rho}\partial_\phi\right)\right),\end{aligned}$$

to $P\Psi = \psi$, we find

$$\begin{aligned}(\hat{p}_1 - C_1)\psi &= -\mathrm{i}\hbar\left(\cos\phi + \mathrm{i}\frac{m\omega\Theta}{2\hbar}\sin\phi\right)\partial_\rho\psi\\ &\quad + \left(-\frac{\hbar|m_l|}{\rho}\sin\phi + \frac{\mathrm{i}\hbar|m_l|}{\rho}\cos\phi + \frac{\Omega}{2\hbar}\rho\sin\phi - \mathrm{i}m\omega\rho\cos\phi\right)\psi.\end{aligned}$$

If we perform the same operation for $\hat{p}_2 - C_2$, we find

$$\begin{aligned}\hat{p}_2 - C_2 &= -\mathrm{i}\hbar\partial_y - \frac{\Omega x}{2\hbar} - \mathrm{i}m\omega\left(y - \mathrm{i}\hbar\frac{\Theta\partial_x}{2\hbar}\right)\\ &= -\mathrm{i}\hbar\left(\sin\phi\,\partial_\rho + \frac{\cos\phi}{\rho}\partial_\phi\right) - \frac{\Omega}{2\hbar}\rho\cos\phi\\ &\quad - \mathrm{i}m\omega\left(\rho\sin\phi - \mathrm{i}\hbar\frac{\Theta}{2\hbar}\left(\cos\phi\,\partial_\rho - \frac{\sin\phi}{\rho}\partial_\phi\right)\right),\end{aligned}$$

which, when applied to $P\Psi = \psi$, gives

$$\begin{aligned}(\hat{p}_2 - C_2)\psi &= -\mathrm{i}\hbar\left(\sin\phi - \mathrm{i}\frac{m\omega\Theta}{2\hbar}\cos\phi\right)\partial_\rho\psi\\ &\quad + \left(\frac{\hbar|m_l|}{\rho}\cos\phi + \frac{\mathrm{i}\hbar|m_l|}{\rho}\sin\phi + \frac{\Omega}{2\hbar}\rho\sin\phi - \mathrm{i}m\omega\rho\cos\phi\right)\psi.\end{aligned}$$

Note that

$$\begin{aligned}\partial_\rho\psi &= \overline{N}\,\rho^{|m_l|-1}e^{\mathrm{i}|m_l|\phi}\left(|m_l| + \frac{M\overline{\omega}_{\Theta,\Omega}}{\hbar}\rho^2\right)e^{-\frac{m\omega}{2\hbar}z^2 - \frac{M\overline{\omega}_{\Theta,\Omega}}{2\hbar}\rho^2}\\ &\quad \times {}_1F_1\left(-n_\rho; |m_l|+1; \frac{M\overline{\omega}_{\Theta,\Omega}}{\hbar}\rho^2\right)H_{n_z}\left(\sqrt{\frac{m\omega}{\hbar}}z\right)\\ &\quad + 2\frac{M\overline{\omega}_{\Theta,\Omega}}{\hbar}\rho\overline{N}\,\rho^{|m_l|}e^{\mathrm{i}|m_l|\phi}e^{-\frac{m\omega}{2\hbar}z^2 - \frac{M\overline{\omega}_{\Theta,\Omega}}{2\hbar}\rho^2}\\ &\quad \times {}_1F_1\left(1-n_\rho; |m_l|+2; \frac{M\overline{\omega}_{\Theta,\Omega}}{\hbar}\rho^2\right)H_{n_z}\left(\sqrt{\frac{m\omega}{\hbar}}z\right).\end{aligned}$$

Therefore, we can write

$$(\hat{p}_1 - C_1)\psi = G_{11}\,{}_1F_1\left(-n_\rho; |m_l|+1; \frac{M\overline{\omega}_{\Theta,\Omega}}{\hbar}\rho^2\right) + G_{12}\,{}_1F_1\left(1-n_\rho; |m_l|+2; \frac{M\overline{\omega}_{\Theta,\Omega}}{\hbar}\rho^2\right)H_{n_z}\left(\sqrt{\frac{m\omega}{\hbar}}z\right),$$

where

$$G_{11} = \overline{N}\left[-\mathrm{i}\hbar\left(\cos\phi + \mathrm{i}\frac{m\omega\Theta}{2\hbar}\sin\phi\right)\left(|m_l| + \frac{M\overline{\omega}_{\Theta,\Omega}}{\hbar}\rho^2\right)\rho^{-1} + \left(-\frac{\hbar|m_l|}{\rho}\sin\phi + \frac{\mathrm{i}\hbar|m_l|}{\rho}\cos\phi + \frac{\Omega}{2\hbar}\rho\sin\phi - \mathrm{i}m\omega\rho\cos\phi\right)\right]\Lambda H_{n_z}\left(\sqrt{\frac{m\omega}{\hbar}}z\right),$$

and

$$G_{12} = -2\mathrm{i}\overline{N}M\overline{\omega}_{\Theta,\Omega}\left(\cos\phi + \mathrm{i}\frac{m\omega\Theta}{2\hbar}\sin\phi\right)\rho\Lambda H_{n_z}\left(\sqrt{\frac{m\omega}{\hbar}}z\right).$$

The symbol Λ is a short-hand for

$$\Lambda = \rho^{|m_l|}e^{\mathrm{i}|m_l|\phi}e^{-\frac{m\omega}{2\hbar}z^2 - \frac{M\overline{\omega}_{\Theta,\Omega}}{2\hbar}\rho^2}.$$

For $\hat{p}_2 - C_2$, we obtain

$$(\hat{p}_2 - C_2)\psi = G_{21}\,{}_1F_1\left(-n_\rho; |m_l|+1; \frac{M\overline{\omega}_{\Theta,\Omega}}{\hbar}\rho^2\right) + G_{22}\,{}_1F_1\left(1-n_\rho; |m_l|+2; \frac{M\overline{\omega}_{\Theta,\Omega}}{\hbar}\rho^2\right),$$

where

$$G_{21} = \overline{N}\left[-\mathrm{i}\hbar\left(\sin\phi - \mathrm{i}\frac{m\omega\Theta}{2\hbar}\cos\phi\right)\left(|m_l| + \frac{M\overline{\omega}_{\Theta,\Omega}}{\hbar}\rho^2\right)\rho^{-1} + \left(\frac{\hbar|m_l|}{\rho}\cos\phi + \frac{\mathrm{i}\hbar|m_l|}{\rho}\sin\phi + \frac{\Omega}{2\hbar}\rho\sin\phi - \mathrm{i}m\omega\rho\cos\phi\right)\right]\Lambda H_{n_z}\left(\sqrt{\frac{m\omega}{\hbar}}z\right),$$

and

$$G_{22} = -2\mathrm{i}\overline{N}M\overline{\omega}_{\Theta,\Omega}\left(\sin\phi - \mathrm{i}\frac{m\omega\Theta}{2\hbar}\cos\phi\right)\Lambda H_{n_z}\left(\sqrt{\frac{m\omega}{\hbar}}z\right).$$

If we proceed similarly for $\hat{p}_3 - C_3 = p_3 - \mathrm{i}m\omega z$, we find

$$(p_3 - \mathrm{i}m\omega z)\psi = (-\mathrm{i}\hbar\partial_z - \mathrm{i}m\omega z)\psi = -\mathrm{i}\hbar\partial_z\psi = G_3\,{}_1F_1\left(-n_\rho; |m_l|+1; \frac{M\overline{\omega}_{\Theta,\Omega}}{\hbar}\rho^2\right),$$

where

$$G_3 = -2\mathrm{i}\sqrt{\frac{m\omega}{\hbar}}\overline{N}\Lambda H_{n_z-1}\left(\sqrt{\frac{m\omega}{\hbar}}z\right).$$

Therefore, we can rewrite the spinor Ψ as

$$i\hbar k\Psi = \begin{pmatrix} G_{11} \\ G_{21} \\ G_3 \\ E \\ m \\ 1 \end{pmatrix} {}_1F_1\left(-n_\rho; |m_l| + 1; \frac{M\overline{\omega}_{\Theta,\Omega}}{\hbar}\rho^2\right)$$
$$+ \begin{pmatrix} G_{12} \\ G_{22} \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} {}_1F_1\left(1 - n_\rho; |m_l| + 2; \frac{M\overline{\omega}_{\Theta,\Omega}}{\hbar}\rho^2\right).$$

4 Concluding Remarks

We have obtained and solved the Galilean DKP wave equation for spin-zero fields in the oscillator potential for a non-commutative (both for coordinates and momenta) space. We obtained the equation by a Lorentz-like approach called 'Galilean covariance' where we begin with manifestly covariant equations in a $(4+1)$-dimensional manifold using light-cone coordinates, and then reduce to the Newtonian 4-dimensional space-time. We have determined the exact wave functions and the corresponding energy levels.

In order to discuss the effects of non-commutativity, notice that Eq. (30) leads to

$$\overline{\omega}_{\Theta=0,\Omega=0} = \omega,$$
$$\overline{\omega}_{\Theta=0,\Omega} = \frac{1}{2m\hbar}\sqrt{4m^2\hbar^2\omega^2 + \Omega^2},$$
$$\overline{\omega}_{\Theta,\Omega=0} = \frac{\omega}{2\hbar}\sqrt{4\hbar^2 + m^2\omega^2\Theta^2}.$$

If we take $\Omega = 0$ and $\Theta = 0$ in Eq. (37), then the energy eigenvalues are given by

$$E = (2n_\rho + |m_l| + n_z)\hbar\omega, \quad (\Omega = 0, \Theta = 0).$$

If we take only $\Theta = 0$ in Eq. (37), this renders the momenta commuting among themselves while keeping the coordinates mutually non-commuting, and the energy eigenvalues become

$$E = (n_z - 1)\hbar\omega + (2n_\rho + |m_l| + 1)\hbar\overline{\omega}_{\Theta=0,\Omega} - \frac{m_l\Omega}{2m}, \quad (\Theta = 0).$$

Instead, if we take only $\Omega = 0$ in Eq. (37), so that we have commuting coordinates and non-commuting momenta in Eq. (5), then the energy is given by

$$E = (n_z - 1)\hbar\omega + (2n_\rho + |m_l| + 1)\hbar\overline{\omega}_{\Theta,\Omega=0} - \frac{1}{2}m_l m\omega^2\Theta, \quad (\Omega = 0).$$

We are currently extending the present work in two directions: to the non-commutative Galilean covariant *Dirac* oscillator (or 'Lévy-Leblond oscillator') and the non-commutative *spin-one* Galilean DKP oscillator. The commutative version of the Galilean Dirac-like equation was examined by Lévy-Leblond in Ref. [54]; its Galilean covariant version is discussed in Ref. [55, 56]. The relativistic Dirac oscillator in a non-commutative phase space has been investigated in Ref. [57]. Finally, it should be interesting to consider the analogy between the oscillator in a non-commutative space and a constant magnetic field in a commutative space, especially since there exist *two* Galilean limits (so-called 'electric' and 'magnetic') of electromagnetism (see [58] and Santos et al. [55, 56]).

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