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ON THE MEASUREMENT OF THE PHASE DISTRIBUTION OF FIELD STATES

C. VALVERDE[*,§], B. BASEIA[*,¶] and J. M. C. MALBOUISSON[†,‡,‖]

[*] *Instituto de Física, Universidade Federal de Goiás,*
74001-970, Goiânia, GO, Brazil
[†] *Instituto de Física, Universidade Federal da Bahia,*
40210-340, Salvador, BA, Brazil
[§] *valverde@unip.br*
[¶] *basilio@fis.ufg.br*
[‖] *jmalboui@ufba.br*

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A proposal for measuring the phase distribution $P(\theta)$ of certain families of field states is presented. This method is based on establishing the connection between the phase distribution and the visibility of the (single-photon) interference in a Mach–Zehnder device. Such a scheme can also be applied to determine relative phases of superposed states.

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1. Introduction

Although coherent[1] and squeezed[2] states are the most widely available pure states of the electromagnetic field, the number and the phase states also occupy an important place in quantum optics, mainly due to their wide use as mathematical tools in this area. These two latter states are complementary in the sense that the number operator $\hat{n}$ and the phase operator $\hat{\phi}$ form a canonically conjugate pair.[3–5]

A property usually studied in the literature is the photon-number distribution $P_n = |\langle n|\Psi\rangle|^2$ which gives a first characterization of a field state $|\Psi\rangle$. The experimental determination of P_n is well known. The knowledge of P_n does not fully characterize a field state $|\Psi\rangle$ at all: actually, there are many examples of distinct field states having the same statistical distribution P_n.[6–8] In such cases, the distinction among these fields can be obtained by looking at their complementary distribution $P(\theta)$. To our knowledge, determining $P(\theta)$ experimentally is not a well established procedure, so a proposal for its implementation is in order.

[‡]Corresponding author.
[§]Present address: Universidade Paulista, 74845-090, Goiânia, GO, Brazil.

Since the appearance of reliable phase states,[3–5] various treatments about the phase distribution $P(\theta)$ have appeared in the literature.[9–16] The Pegg–Barnett (PB) truncated phase states,

$$|\theta_m\rangle = \frac{1}{\sqrt{s+1}} \sum_{n=0}^{s} \exp(in\theta_m)|n\rangle\,, \tag{1}$$

with

$$\theta_m = \theta_0 + m\left(\frac{2\pi}{s+1}\right), \qquad m = 0, 1, 2, \ldots, s\,, \tag{2}$$

and the PB phase operator,

$$\hat{\phi} = \sum_{m=0}^{s} \theta_m |\theta_m\rangle\langle\theta_m|\,, \tag{3}$$

were introduced in Refs. 3–5; the set of states $\{|\theta_m\rangle\}$ forms a complete orthogonal set, $\sum_{m=0}^{s} |\theta_m\rangle\langle\theta_m| = \hat{1}$ and $\langle\theta_m|\theta_{m'}\rangle = \theta_{m,m'}$, in the truncated Hilbert space of dimension $s+1$, i.e. $\hat{\phi}$ is a Hermitian operator. The ideal phase state is obtained in the limit $s \to \infty$. For an arbitrary pure-state $|\Psi\rangle$, $P(\theta)$ is defined as

$$P(\theta) = \lim_{s\to\infty} \frac{s+1}{2\pi} |\langle\theta_m|\Psi\rangle|^2 = \frac{1}{2\pi} \sum_{n,n'=0}^{\infty} \rho_{n,n'} \exp[-i(n-n')\theta]\,, \tag{4}$$

where $\rho_{n,n'} = \langle n|\Psi\rangle\langle\Psi|n'\rangle$ are the matrix elements of the density operator describing the field state in the number basis.

In a recent paper,[17] a proposal to measure the phase distribution $P(\theta)$ of an arbitrary field state was presented. The situation there was concerned with a field in a running wave and required an auxiliary field named the Reciprocal Binomial State which had a special characteristic. A proposal to generate such an auxiliary state was presented later,[18] a result which is also crucial for quantum lithography.[19]

In this report, we will present an alternative procedure to measure the phase distribution $P(\theta)$ of certain families of field states. The method employs single-photon interference in a Mach–Zehnder interferometer and takes advantage of previous theoretical results.[20,21] In Sec. 2, we present the experimental arrangement and briefly discuss the phase shift measurement using such a device. Section 3 is concerned with the calibration of the apparatus to determine the phase distribution $P(\theta)$. In Sec. 4, an application of this scheme to determine the relative phase of superposed states is presented; while Sec. 4 contains our final remarks.

2. Phase Measurement via Single-Photon Interference

The schematic arrangement for our proposed experiment to measure the phase distribution $P(\theta)$ of a field state $|\Psi\rangle$, consisting of a single-photon Mach–Zehnder interferometer with one arm connected to a (nonlinear) Kerr medium, is shown in Fig. 1. The visibility v of the interference pattern will depend on the accuracy of

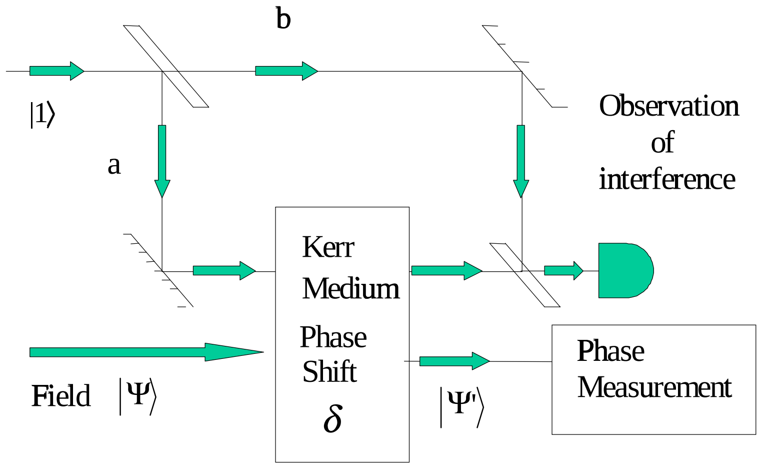

Fig. 1. Single-photon interferometer including a Kerr medium for QND measurement of the photon number in arm a. Accurate determination of the phase shift in the field $|\Psi\rangle$ destroys the interference pattern.

the measurement of the phase shift δ produced upon the probe field $|\Psi\rangle$ entering the additional arm c and traversing the nonlinear medium.[20] It reads

$$v = |\langle\Psi|e^{i\hat{n}\delta}|\Psi\rangle| \tag{5}$$

where $e^{i\hat{n}\delta}|\Psi\rangle = |\Psi'\rangle$ is the state describing the output emerging from the nonlinear medium inserted on the arm c. In the number basis, $|\Psi\rangle = \sum_n c_n|n\rangle$, one has

$$v = \left|\sum_n P_n e^{in\delta}\right| . \tag{6}$$

Since the interaction modifying the state depends only on the operator $\exp(i\hat{n}\delta)$, it does not change the number of photons (0 or 1) traversing the arm b, so the scheme constitutes a quantum nondemolition measurement (QND). Measuring the phase shift δ in the field $|\Psi\rangle$ with high accuracy destroys the interference pattern,[20] in agreement with a Bohr statement.[22]

However, when the phase of the field state $|\Psi\rangle$ does not have a good definition, its phase shift δ cannot be obtained with high accuracy.[21] In such a case, the interference pattern is just partially destroyed. So, there is a correspondence between the visibility v of the interference pattern and the phase definition of a field state. In Ref. 21, we discussed the Heisenberg limit concerning the accuracy of a phase measurement, by taking advantage of the procedure of Refs. 20 and 23. Here, we will apply this scheme to the determination of the phase distribution, for certain families of field states, as discussed in the next section.

3. "Calibration" of the Apparatus

The point to be solved is: how to establish a correspondence between the phase distribution $P(\theta)$ and the visibility v of the interference pattern? Clearly, there is no universal relation between v and $P(\theta)$ for an arbitrary state $|\Psi\rangle$: the visibility is specified by the photon-number distribution and the dispersive interaction in the Kerr medium; while $P(\theta)$ requires all the elements of the density matrix in the number basis to be known. Nevertheless, for certain families of states characterized by one parameter, a correspondence between v and $P(\theta)$ may be established.

To this end, let us consider a probe field which interpolates between a number state $|N\rangle$ and a truncated phase state $|\theta_m^{(S)}\rangle$, both pertaining to the $(S+1)$-dimensional Hilbert space, as follows:

$$|\Psi(\xi)\rangle = \eta\left[\sqrt{\xi}|N\rangle + \sqrt{1-\xi}|\theta_m^{(S)}\rangle\right], \tag{7}$$

where the normalization factor is given by

$$\eta = \left[1 + 2\sqrt{\frac{\xi(1-\xi)}{S+1}}\cos(N\theta_m^{(S)})\right]^{-\frac{1}{2}}. \tag{8}$$

Note that for $\xi = 1$ and $\xi = 0$, the state $|\Psi(\xi)\rangle$ coincides with the number state $|N\rangle$ and the PB phase state $|\theta_m^{(S)}\rangle$, respectively. When $\xi \in (0,1)$ the state $|\Psi(\xi)\rangle$ is intermediate between $|N\rangle$ and $|\theta_m^{(S)}\rangle$. In the limit of very large S, in the extreme of $\xi = 0$ ($\xi = 1$), the state $|\Psi(\xi)\rangle$ has a well-defined (random) phase; probe fields in the first (second) extreme will correspond to $v = 0$ ($v = 1$).

Now, by substituting Eqs. (7) and (8) into Eq. (5), the visibility v in the interferometer, when the probe field is one of the states in Eq. (7), is given by

$$v = \left\{A^2 + B^2 + 2AB\cos\left[\left(\frac{S}{2} - N\right)\delta\right]\right\}^{\frac{1}{2}} \tag{9}$$

where

$$A = \eta^2\left[\xi + 2\sqrt{\frac{\xi(1-\xi)}{S+1}}\cos(N\theta_m^{(S)})\right], \tag{10}$$

$$B = \eta^2\left[\frac{1-\xi}{S+1}\frac{\sin[(S+1)\frac{\delta}{2}]}{\sin(\frac{\delta}{2})}\right]. \tag{11}$$

In Fig. 2, we plot the visibility as a function of the parameter ξ for a given family of states (specified by S, N, θ_0 and m), for some values of δ. We notice that the behavior of v versus ξ depends crucially on the value of δ. For families of states of the kind we are considering, however, the distinction among curves for different values of δ tends to disappear for large values of S.

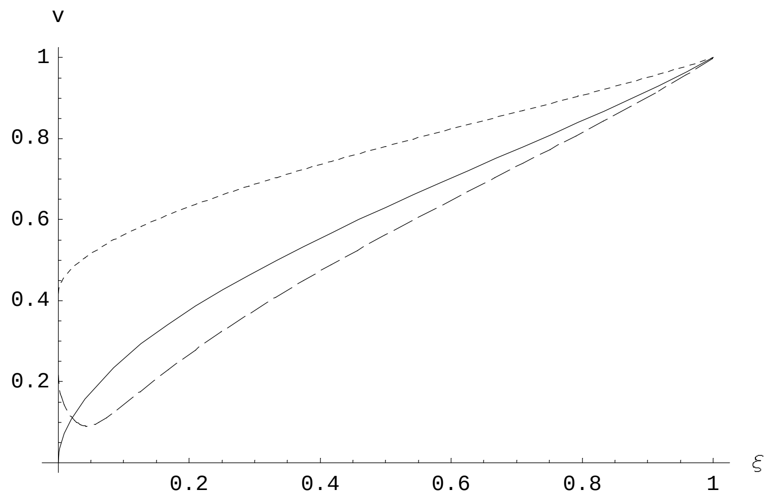

Fig. 2. Visibility v as a function of ξ, for the family of states $|\Psi(\xi)\rangle$ with $S = 7$, $N = 4$, $\theta_0 = 0$ and $m = 0$, for different values of the phase shift δ: $\pi/2$ (full line), $\pi/3$ (dashed line) and $\pi/6$ (dotted line).

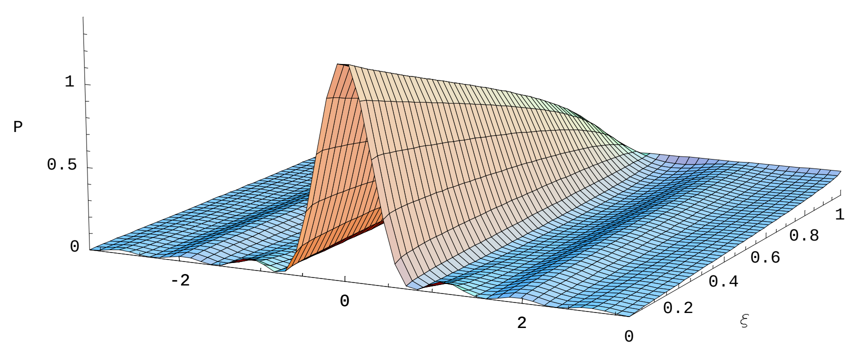

Fig. 3. Phase distribution as a function of θ and ξ, for the same family of states as in Fig. 2.

On the other hand, calculating the phase distribution $P(\theta)$ for the same family $|\Psi(\xi)\rangle$, we obtain

$$P(\theta, \xi) = \frac{\eta^2}{2\pi}\left[\xi + 2\sqrt{\frac{\xi(1-\xi)}{S+1}}F(S,\beta)\cos\left(\frac{S\beta}{2} + N\theta\right) + \frac{1-\xi}{S+1}[F(S,\beta)]^2\right], \quad (12)$$

where

$$F(S,\beta) = \frac{\sin[(S+1)\frac{\beta}{2}]}{\sin(\frac{\beta}{2})} \quad (13)$$

and $\beta = \theta_m^{(S)} - \theta$. Figure 3 shows $P(\theta;\xi)$ for the same states $|\Psi(\xi)\rangle$ as in Fig. 2.

These results allow us to establish a correspondence between $P(\theta)$ and v. In fact, since the visibility can be taken as a monotonically increasing function of ξ by an appropriate choice of δ, an estimation of v leads to the phase distribution $P(\theta)$ for the state defined by the corresponding value of ξ. The correspondence between v and $P(\theta)$ is then obtained from Eqs. (9) and (12) through the elimination of the parameter ξ. Accordingly, for the family of states $|\Psi(\xi)\rangle$ given in Eq. (7), by properly adjusting the phase shift δ, the visibility v of the interference pattern determines the phase distribution $P(\theta)$. Notice that (contrary to Refs. 20 and 23 where δ is a wanted parameter), δ is a choice parameter here, relevant to the setting of the experimental apparatus.

The same procedure can be applied to other families of field states. For example:

(i) A family of coherent states, $|\Psi(\xi)\rangle = |\alpha\rangle$ with $\alpha = \xi \exp(i\phi)$. In this case, taking $\delta \ll 1$, we obtain $v \cong \exp(-\xi^2\delta^2/2)$, and, for $\xi = |\alpha| \gg 1$, $P(\theta, \alpha) \cong (2\xi^2/\pi)^{1/2} \exp[-2\xi^2(\phi - \theta)^2]$. Hence, a new connection between v and $P(\theta)$ can be obtained for this family of field states.

(ii) A family of squeezed states, $|\Psi(\xi)\rangle = |z, \alpha\rangle$, with fixed α and $\xi = |z|$, etc.

We call these procedures (in which we use known one-parameter families of field states) to establish the connection between $P(\theta)$ and v by adjusting the phase shift δ, "calibrations" of the Mach–Zehnder interferometer. Naturally, the method does not work for arbitrary field states, since it requires some knowledge about the class of probe states entering the apparatus. Once this previous knowledge is given, the present scheme turns out to be a good tool to determine $P(\theta)$. As mentioned before, a proposal for the determination of $P(\theta)$ for traveling waves in arbitrary states has been studied in the literature.[17] However, this method requires the use of the reciprocal-binomial state whose generation has only been discussed for trapped states.[18]

4. Relative Phase Measurement

The measurement of the visibility in the Mach–Zehnder apparatus can also be used to determine the relative classical phase between components of a superposition of coherent states of equal intensity. Consider the state

$$|\Psi\rangle = \eta(|\alpha\rangle + |e^{i\phi}\alpha\rangle)\,, \tag{14}$$

where α is real (for simplicity) and $\eta = [2 + 2e^{-\alpha^2(1-\cos\phi)}\cos(\alpha^2\sin\phi)]^{-1/2}$ is the normalization factor. If the state (14) is taken as the probe field injected into the c arm of the apparatus, then the measured visibility is given by

$$\begin{aligned} v(\delta;\phi;\alpha) = {} & [2e^{-\alpha^2} + 2e^{\alpha^2\cos\phi}\cos(\alpha^2\sin\phi)]^{-1} \\ & \times \{4e^{2\alpha^2\cos\delta} + e^{2\alpha^2\cos(\delta+\phi)} + e^{2\alpha^2\cos(\delta-\phi)} \\ & + 4e^{\alpha^2[\cos\delta+\cos(\delta+\phi)]}\cos[\alpha^2(\sin(\delta+\phi) - \sin\delta)] \\ & + 4e^{\alpha^2[\cos\delta+\cos(\delta-\phi)]}\cos[\alpha^2(\sin(\delta-\phi) - \sin\delta)] \\ & + 2e^{2\alpha^2\cos\delta\cos\phi}\cos(2\alpha^2\cos\delta\sin\phi)\}^{1/2}\,. \end{aligned} \tag{15}$$

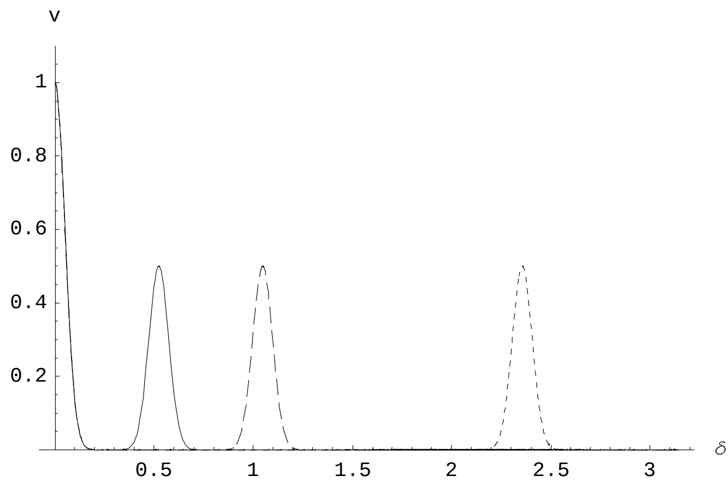

Fig. 4. Visibility v as a function of the phase shift δ for states in Eq. (14), taking $\alpha = 20$, with $\phi = \pi/6$ (full line), $\phi = \pi/3$ (dashed line) and $\phi = 3\pi/4$ (dotted line). The peak at $\delta = 0$ coincides for the three cases, as it should.

One notices immediately that the visibility is periodic in δ with a period of 2π. Variations of the phase shift δ can be obtained by changing the extension of the non-linear Kerr medium where the auxiliary field travels. On the other hand, $v(\delta;\phi;\alpha)$ is invariant under the transformation $\phi \to 2\pi - \phi$, which means that the profile of v versus δ (with fixed ϕ) for $\delta \in [\pi, 2\pi]$ is the specular reflection on the line $\delta = \pi$ of that corresponding to the segment $[0, \pi]$, and vice-versa. In other words, the visibility does not distinguish ϕ from $2\pi - \phi$.

For very large intensities of the component states, $\alpha \gg 1$, the plot of v versus δ shows, besides the intrinsic peak for $\delta = 0$ (since as $\delta \to 0$, $v \to 1$ for any state), a well-defined peak of height $1/2$ when $\delta = 0$ (but with ϕ not too close to π) has illustrated in Fig. 4. We see then that, with the limitation imposed by the symmetry mentioned above, the analysis of the profile $v(\delta)$ allows one to determine the relative classical phase ϕ between the two coherent states. When ϕ approaches π, the two peaks tend to merge into one another, forming a single peak with height reaching 1 exactly for $\phi = 0$. Phase shifts of the order of π were achieved in recent experiments.[24]

5. Conclusions

We have discussed a new proposal to measure the phase distribution $P(\theta)$ of one-parameter families of field states. The scheme employs a single-photon Mach–Zehnder interferometer, which has in one arm, a nonlinear medium that couples a probe field $|\Psi(\xi)\rangle$ (in an auxiliary arm c) with a single photon (in arm a). The dispersive interaction in the nonlinear medium causes a phase shift δ in the field $|\Psi(\xi)\rangle$. This phase shift can be measured with high accuracy when the field $|\Psi(\xi)\rangle$

has good phase definition: the ideal extreme case being the field in a phase state.[21] In this case, the interference pattern measured in the Mach–Zehnder device is destroyed. On the contrary, complete destruction of the interference pattern is no longer observed when the field $|\Psi(\xi)\rangle$ does not have good phase definition: the extreme example being given by a field prepared in a number state which has a random phase. The method here consists of establishing a one-to-one correspondence between the phase distribution $P(\theta)$ and the visibility v of the interference pattern. Such a calibration allows one to obtain the phase distribution from the visibility. It is worth mentioning that, as also occurs for photon number distribution P_n,[6–8] the determination of $P(\theta)$ does not characterize completely the state $|\Psi(\xi)\rangle$. In fact, in setting $\hat{\rho} = |\Psi(\xi)\rangle\langle\Psi(\xi)|$, we note that $P(\theta)$ only defines the diagonal part of the matrix $\langle\theta|\hat{\rho}|\theta'\rangle$. In another interesting application of the present scheme, we have shown how to determine the relative classical phase between two superposed coherent states, obtained from the maximum of the measured visibility v plotted as a function of the phase shift δ (cf. Fig. 4). The accuracy in this determination increases when $|\alpha|$ becomes large, yielding the Kerr medium to play an efficient role in the process. Needless to say, the method proposed only works for traveling fields.

Finally, it is worth observing that there are various proposals in the literature to obtain the Wigner function characterizing the system completely (these procedures being usually named quantum state tomography).[25–29] Alternatively, there are also various proposals presented in the literature allowing one to reconstruct the wave-function describing a system. Such procedures are usually named quantum state endoscopy.[30–33] These methods provide all information about the system without any *a priori* information. However, concerning with feasibility, and contrary to our case, these methods are very sophisticated requiring the use of high technologies (high-Q cavities, use of very low temperatures to avoid environmental noises and decoherence effects, high efficiency detectors, errors arising from fluctuating parameters, etc). So, although the complete knowledge of a field state can be obtained in suitable situations, the study of particular properties of a field state is not useless. That is the reason why various techniques have been developed to determine particular properties of field states: e.g. measuring the variance of quadrature operators to check the occurrence of squeezing, measuring second-order correlation functions to check the occurrence of anti-bunching; and many other properties, the phase distribution $P(\theta)$ being an example of them.

Acknowledgments

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