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Hitchin–Thorpe inequality and Kaehler metrics for compact almost Ricci soliton

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Abstract The purpose of this paper is to prove a Hitchin–Thorpe inequality for a four-dimensional compact almost Ricci soliton. Moreover, we prove that under a suitable integral condition, a four-dimensional compact almost Ricci soliton is isometric to standard sphere. Finally, we prove that under a simple condition, a four-dimensional compact Ricci soliton with harmonic self-dual part of Weyl tensor is either isometric to a standard sphere $\mathbb{S}^4$ or is Kaehler–Einstein.

Keywords Ricci soliton · Almost Ricci soliton · Kaehler metrics · Hitchin–Thorpe inequality

Mathematics Subject Classification (2000) Primary: 53C25, 53C20, 53C21 · Secondary: 53C65

1 Introduction and Statement of the results

The study of an almost Ricci soliton was introduced in a recent paper due to Pigola et al. [13], where essentially the authors modified the definition of Ricci soliton by adding the condition

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on the parameter λ to be a variable function. More precisely, we say that a Riemannian manifold (M^n, g) is an almost Ricci soliton, if there exists a complete vector field X and a smooth soliton function $\lambda : M^n \to \mathbb{R}$ satisfying

$$\mathrm{Ric} + \frac{1}{2}\mathcal{L}_X \mathrm{g} = \lambda \mathrm{g}, \tag{1.1}$$

where Ric and $\mathcal{L}$ stand, respectively, for the Ricci tensor and the Lie derivative. We shall refer to this equation as the fundamental equation of an almost Ricci soliton $\left(M^n, g, X, \lambda\right)$. It will be called *expanding*, *steady* or *shrinking*, respectively, if $\lambda < 0, \lambda = 0$ or $\lambda > 0$. Otherwise, it will be called *indefinite*. When the vector field X is a gradient of a smooth function $f : M^n \to \mathbb{R}$, the manifold will be called a gradient almost Ricci soliton. In this case, the preceding equation turns out to be

$$\mathrm{Ric} + \nabla^2 f = \lambda g, \tag{1.2}$$

where $\nabla^2 f$ stands for the Hessian of f.

Moreover, when either the vector field X is trivial, or the potential f is constant, the almost Ricci soliton will be called *trivial*, otherwise it will be a *nontrivial* almost Ricci soliton. We notice that when $n \geq 3$ and X is a Killing vector field, an almost Ricci soliton will be a Ricci soliton, since in this case, we have an Einstein manifold, from which we can apply Schur's lemma to deduce that λ is constant. Taking into account that the soliton function λ is not necessarily constant, certainly comparison with soliton theory will be modified. In particular, the rigidity result contained in Theorem 1.3 of [13] indicates that almost Ricci solitons should reveal a reasonably broad generalization of the fruitful concept of classical soliton. In fact, we refer the reader to [13] to see some of these changes. Ricci solitons model the formation of singularities in the Ricci flow, and they correspond to self-similar solutions, i.e., they are stationary points of this flow in the space of metrics modulo diffeomorphisms and scalings; see [10] for more details. Thus, classifying Ricci solitons or understanding their geometry is definitely an important issue.

In the direction of understanding the geometry of almost Ricci soliton, Barros and Ribeiro Jr proved in [1] that a compact almost Ricci soliton with nontrivial conformal vector field is isometric to a Euclidean sphere. In the same paper, they proved an integral formula for compact case, which was used to prove several rigidity results, for more details see [1]. Pigola et al. proved a classification to gradient almost Ricci soliton under Einstein assumption, see Theorem 1.3 in [13]. In [5], Catino proved that a locally conformally flat gradient almost Ricci soliton, around any regular point of f, is locally a warped product with $(n-1)$-dimensional fibers of constant sectional curvature.

In the compact case, a simple example of almost Ricci soliton appeared in [1]. It was built over the standard sphere $\left(\mathbb{S}^n, g_0\right)$ endowed with the conformal vector field $X = a^\top$, where a is a fixed vector in $\mathbb{R}^{n+1}$ and $a^\top$ stands for its orthogonal projection over $T\mathbb{S}^n$. We notice that $a^\top$ is the gradient of the height function h_a; for more details see the quoted paper. In the non-compact case, an another example appeared in [2]. It was obtained in a warped product manifold $M^{n+1} = \mathbb{R} \times_{\cosh t} \mathbb{S}^n$ with metric $g = \mathrm{d}t^2 + \cosh^2 t g_0$, where g_0 is the standard metric of $\mathbb{S}^n$. More precisely, taking $\left(M^{n+1}, g, \nabla f, \lambda\right)$, where $f(x,t) = \sinh t$ and $\lambda(x,t) = \sinh t + n$, it is easy to check, by using Lemma 1.1 of [13], that $\left(M^{n+1}, g, \nabla f, \lambda\right)$ is an almost Ricci soliton.

We also detach that in [14] Perelman proved that every compact Ricci soliton is gradient. Recently, Pigola et al. [13] asked under which conditions a compact almost Ricci soliton is necessarily gradient. To answers this problem Barros et al. [3] proved that every compact almost Ricci soliton with constant scalar curvature is gradient.

1.1 Hitchin–Thorpe inequality for almost Ricci soliton

From now on, we consider M^4 be a compact-oriented four-dimensional manifold with Euler characteristic χ, signature τ, volume V and volume element $d\mu$. Hitchin [11] and Thorpe [17] have proved, independently, that if M^4 is an Einstein manifold, then $\chi \geq \frac{3|\tau|}{2}$. This inequality is known as Hitchin–Thorpe, and it has various geometric implications. Since almost Ricci soliton is natural generalization to Einstein metrics, it is very interesting to obtain a Hitchin–Thorpe inequality type to almost Ricci soliton. Indeed, by using the previous notation, we obtain the following theorem.

Theorem 1 *Let $\left(M^4,\ g,\ \nabla f,\ \lambda\right)$ be a four-dimensional compact almost Ricci soliton with positive scalar curvature R.*

(1) *If*

$$\int_M R^2 d\mu \leq 6 \int_M \lambda R d\mu,$$

then $\chi \geq \frac{3\tau}{2}$. In particular, if λ is constant and $\int_M R^2 d\mu \leq 24\lambda^2 V$, then $\chi \geq \frac{3|\tau|}{2}$.

(2) *If (M^4, g) is Kaehlerian, then*

$$2\chi + 3\tau = \frac{1}{2\pi^2} \int_M \lambda R d\mu.$$

We point out that Theorem 1 can be viewed as a generalization of Ma's theorem [12], which was proved to λ constant.

In [16], Seshadri proved that a compact-oriented four-dimensional manifold with Euler characteristic χ, Weyl tensor W and positive scalar curvature satisfies the following inequality

$$8\pi^2(\chi - 2) \leq \int_M |W|^2 d\mu. \tag{1.3}$$

In fact, this result was already contained in the proofs of the results of M. Gursky in [9]. The equality holds in (1.3) if and only if M^4 is isometric to a standard sphere $\mathbb{S}^4$. By using Seshadri-Gursky theorem, we obtain the following rigidity theorem.

Theorem 2 *Let $\left(M^4,\ g,\ \nabla f,\ \lambda\right)$ be a four-dimensional compact gradient almost Ricci soliton with positive scalar curvature R. If*

$$\int_M R^2 d\mu \leq 6 \int_M \lambda R d\mu - 192\pi^2,$$

then M^4 is isometric to a standard sphere $\mathbb{S}^4$.

As a consequence of Theorem 2, we deduce the following result.

Corollary 1 *Let $\left(M^4,\ g,\ X,\ \lambda\right)$ be a four-dimensional compact Ricci soliton. Then, M^4 is isometric to a standard sphere $\mathbb{S}^4$ provided*

$$\int_M R^2 d\mu \leq 24\left(\lambda^2 V - 8\pi^2\right),$$

where R stands for the scalar curvature of M^4.

Proceeding, we recall that Polombo [15] proved that if M^4 is a compact-oriented four-dimensional Riemannian manifold with (2/3)-pinched Ricci curvature, then the Euler characteristic of M^4 and its signature satisfies the Hitchin–Thorpe inequality. Moreover, in [18], Xin proved that if M^4 is a compact-oriented four-dimensional Kaehler manifold with $(\sqrt{2}/2)$-pinched Ricci curvature, then the Euler characteristic of M^4 and its signature satisfies the Hitchin–Thorpe inequality. The next result improves Polombo's theorem and Xin's theorem. More precisely, we have the following result.

Theorem 3 *Let M^4 be a four-dimensional compact manifold with scalar curvature R and Ricci curvature Ric.*

(1) *If* $\mathrm{Ric} \geq \rho > 0$ *and* $R \leq 6\rho$ *or* $\mathrm{Ric} \leq -\rho < 0$ *and* $R \geq -6\rho$, *then* $\chi \geq \frac{3|\tau|}{2}$.
(2) *Let M^4 Kaehlerian endowed with its natural orientation. If* $\mathrm{Ric} \geq \rho > 0$ *and* $R \leq (6+2\sqrt{3})\rho$ *or* $\mathrm{Ric} \leq -\rho < 0$ *and* $R \geq -(6+2\sqrt{3})\rho$, *then* $\chi \geq -\frac{3\tau}{2}$.

1.2 Harmonic self-dual Weyl tensor in four-dimensional manifolds

The bundle of two forms of M^4 splits $\Lambda^2 M = \Lambda^+ \oplus \Lambda^-$ into $\pm$-eigenspaces of the Hodge $*$-operator. The Weyl tensor W is an endomorphism of $\Lambda^2 M$ such that

$$W = W^+ \oplus W^-,$$

where $W^\pm : \Lambda^\pm \longrightarrow \Lambda^\pm$ are, respectively, the self-dual and anti-self-dual parts of W. Viewing W^+ as a tensor of type (0, 4), we say that W^+ is harmonic if $\delta W^+ = 0$, where δ is the formal divergence defined for any (0, 4)-tensor T by

$$\delta T(X_1, X_2, X_3) = -\mathrm{trace}_g\{(Y, Z) \mapsto \nabla_Y T(Z, X_1, X_2, X_3)\},$$

where g is the metric of M^4.

Let $\lambda_+ \leq \mu_+ \leq \nu_+$ be the eigenvalues of W^+. It is known that if M^4 is Kaehlerian endowed with natural orientation, then W^+ has eigenvalues $R/6$, $-R/12$ and $-R/12$, where R is the scalar curvature of M^4 and so $\mu_+ = -R/12$ is a necessary condition for the metric of a 4-manifold to be Kaehlerian. Inspired by ideas of Derdzinsky developed in [7] we obtain the following theorem.

Theorem 4 *Let M^4 be a compact-oriented four-dimensional manifold with negative scalar curvature R and harmonic tensor W^+. Let $\lambda_+ \leq \mu_+ \leq \nu_+$ be the eigenvalues of W^+. If $\mu_+ \geq -R/12$ in M^4, then M^4 or the twofold covering of M^4 admits a Kaehler metric.*

Notice that if M^4 is a four-dimensional Einstein manifold, then M^4 has harmonic tensor W^+ (for more details see 16.65 in [4]). Therefore, it is very interesting to know when the reverse holds. Recently, Barros et al. [2] proved that an n-dimensional locally conformally flat compact almost Ricci soliton $\left(M^n, g, \nabla f, \lambda\right)$ satisfying a suitable integral condition must be isometric to a Euclidean sphere $\mathbb{S}^n$. The next result gives a condition for a compact almost Ricci soliton with harmonic tensor W^+ to be Kaehler–Einstein. More precisely, we have the following theorem.

Theorem 5 *Let $\left(M^4, g, \nabla f, \lambda\right)$ be a four-dimensional compact gradient almost Ricci soliton with harmonic tensor W^+. If W^+ has constant norm equal or less than $\frac{\sqrt{6}}{3V}\int_M \lambda \mathrm{d}\mu$, then either $W^+ = 0$ or $|W^+| = \frac{\sqrt{6}}{3V}\int_M \lambda \mathrm{d}\mu$. In the last case, M^4 is Kaehler–Einstein*

In [6] was proved that any four-dimensional complete gradient shrinking Ricci soliton with bounded curvature and $W^+ = 0$ must be isometric to a finite quotient of $\mathbb{R}^4$, $\mathbb{S}^3 \times \mathbb{R}$, $\mathbb{S}^4$

or $\mathbb{CP}^2$. Here, as consequence of Theorem 5, we deduce a rigidity result for a Ricci soliton with harmonic self-dual part of Weyl tensor. More exactly, we have the following result.

Corollary 2 *Let* $\left(M^4, g, \lambda\right)$ *be a four-dimensional compact Ricci soliton with harmonic tensor* W^+. *If* W^+ *has constant norm, then either* M^4 *is isometric to sphere* $\mathbb{S}^4$ *or* M^4 *is Kaehler–Einstein.*

2 Proof of the results

Throughout this section, we collect some definitions and results that will be useful in the proofs of our results. We start by presenting the following lemma.

Lemma 1 *Let* $\left(M^4, g, \nabla f, \lambda\right)$ *be a four-dimensional compact gradient almost Ricci soliton. Then*

$$\int_M |\mathrm{Ric}|^2 \mathrm{d}\mu = \frac{1}{2}\int_M (R^2 - 2\lambda R)\mathrm{d}\mu. \tag{2.1}$$

Proof Since $\left(M^4, g, \nabla f, \lambda\right)$ is a compact gradient Ricci almost soliton, by using equation (2.15) of [13], we have

$$\frac{1}{2}\Delta R - \frac{1}{2}\langle \nabla R, \nabla f\rangle = \lambda R - |\ \mathrm{Ric}\ |^2 + 3\Delta\lambda. \tag{2.2}$$

Applying Green's formula in the previous expression we deduce

$$\int_M \langle \nabla R, \nabla f\rangle \mathrm{d}\mu = -\int_M R\Delta f \mathrm{d}\mu.$$

Comparing the last equation and the first item of Proposition 1 in [1] we obtain (2.1), which finishes the proof of the lemma. □

2.1 Proof of Theorem 1

Proof First, let $W^{\pm}$ be the self-dual and anti-self-dual parts of the Weyl tensor W of M^4, respectively. Then, the Euler characteristic χ of M^4 and its signature τ satisfies

$$8\pi^2\chi = \int_M \left(|W^+|^2 + |W^-|^2 + \frac{R^2}{24} - \frac{1}{2}|B|^2\right)\mathrm{d}\mu \tag{2.3}$$

and

$$12\pi^2\tau = \int_M \left(|W^+|^2 - |W^-|^2\right)\mathrm{d}\mu, \tag{2.4}$$

where B is the traceless of Ric tensor of M^4 given by $B = \mathrm{Ric} - \frac{R}{4}g$. In particular, $|B|^2 = |\mathrm{Ric}|^2 - \frac{R^2}{4}$. Follows from (2.3) and (2.4) that

$$4\pi^2(2\chi \pm 3\tau) \geq \int_M \left(\frac{R^2}{24} - \frac{1}{2}|B|^2\right)\mathrm{d}\mu. \tag{2.5}$$

By using $|B|^2 = |\mathrm{Ric}|^2 - \frac{R^2}{4}$ we obtain

$$4\pi^2(2\chi \pm 3\tau) \geq \int_M \Big(\frac{R^2}{6} - \frac{|\,Ric\,|^2}{2}\Big)\mathrm{d}\mu.$$

Now, applying Lemma 1 in the previous inequality we deduce

$$4\pi^2(2\chi \pm 3\tau) \geq \int_M \Big(-\frac{R^2}{12} + \frac{\lambda R}{2}\Big)\mathrm{d}\mu,$$

hence, the first item follows.

Next, since (M^4, g) is Kaehlerian, it is well known that $|W^+| = \frac{R^2}{24}$, where W^+ is the self-dual part of the Weyl tensor. Comparing (2.3) with (2.4) we obtain

$$8\pi^2\chi + 12\pi^2\tau = \int_M \Big(|W^+|^2 + \frac{R^2}{24} - \frac{1}{2}|B|^2\Big)\mathrm{d}\mu.$$

On the other hand, using again that $|B|^2 = |\mathrm{Ric}|^2 - \frac{R^2}{4}$, we deduce

$$8\pi^2\chi + 12\pi^2\tau = \int_M \frac{R^2}{4}\mathrm{d}\mu - \frac{1}{2}\int_M |\mathrm{Ric}|^2\mathrm{d}\mu,$$

therefore, we may use again Lemma 1 to conclude the proof of the theorem. □

2.2 Proof of Theorem 2

Proof First, we assume that $\big(M^4,\ g,\ \nabla f,\ \lambda\big)$ is a four-dimensional compact gradient almost Ricci soliton with positive scalar curvature R satisfying $\int_M R^2\mathrm{d}\mu \leq 6\int_M \lambda R\mathrm{d}\mu - 192\pi^2$. Now, we suppose by contradiction that M^4 is not isometric to a standard sphere. Therefore, we may use the Gursky-Seshadri theorem to infer

$$8\pi^2(\chi - 2) > \int_M |W|^2\mathrm{d}\mu = \int_M \Big(|W^+|^2 + |W^-|^2\Big)\mathrm{d}\mu.$$

Next, comparing equation (2.3) with the previous inequality we obtain

$$\int_M |B|^2\mathrm{d}\mu > \frac{1}{24}\int_M R^2\mathrm{d}\mu - 16\pi^2.$$

On the other hand, since $|B|^2 = -\frac{R^2}{4} + |\mathrm{Ric}|^2$, we may use (2.1) to deduce

$$\int_M R^2\mathrm{d}\mu > 6\int_M \lambda R\mathrm{d}\mu - 192\pi^2,$$

which gives a contradiction. So, we have finished the proof of the theorem. □

2.3 Proof of Corollary 1

Proof Since $\left(M^4,\ g,\ X,\ \lambda\right)$ be a four-dimensional compact Ricci soliton, we may invoke Perelman's Theorem [14] to deduce that the Ricci soliton is gradient. Moreover, using Proposition 3.4 of [8], we conclude that its scalar curvature is positive. Therefore, we may apply Theorem 2 to conclude the proof of the corollary. □

2.4 Proof of Theorem 3

Proof We assume that M^4 satisfies $\mathrm{Ric} \geq \rho > 0$ and $R \leq 6\rho$. Let $r_1 \leq r_2 \leq r_3 \leq r_4$ be the eigenvalues of the Ricci tensor of M^4. Then $R = \sum r_i$ and

$$|B|^2 = \sum r_i^2 - R^2/4 = R^2 - R^2/4 - 2\sum_{i<j} r_i r_j = 3R^2/4 - 2\sum_{i<j} r_i r_j. \tag{2.6}$$

Now,

$$(r_i - \rho)(r_j - \rho) \geq 0 \tag{2.7}$$

for all $i < j$. From this it follows that

$$\sum_{i<j} r_i r_j \geq -6\rho^2 + 3\rho R. \tag{2.8}$$

By using (2.6), (2.8) and (2.5) we obtain

$$4\pi^2(2\chi \pm 3\tau) \geq \int_M \Big(-\frac{R^2}{3} + 3\rho R - 6\rho^2\Big)\mathrm{d}\mu = -\frac{1}{3}\int_M (R - 3\rho)(R - 6\rho)\mathrm{d}\mu \geq 0, \tag{2.9}$$

since $R \leq 6\rho$ and $\mathrm{Ric} \geq \rho$ we have $6\rho \geq R \geq 4\rho > 3\rho$, which implies $\chi \geq \frac{3|\tau|}{2}$.

On the other hand, we consider $\mathrm{Ric} \leq -\rho < 0$ and $R \geq -6\rho$. Then $(r_i + \rho)(r_j + \rho) \geq 0$ for all $i < j$, where $r_1 \leq .... \leq r_4$ are the eigenvalues of the Ricci tensor of M^4. In this case, the proof is analogous. So, we conclude the proof of the first item.

Next, let M^4 be Kaehlerian endowed with its natural orientation. Since $|W^+|^2 = \frac{R^2}{24}$, we can use (2.3) and (2.4) to obtain

$$4\pi^2(2\chi + 3\tau) = \int_M \Big(\frac{R^2}{12} - \frac{1}{2}|B|^2\Big)\mathrm{d}\mu. \tag{2.10}$$

Now, let $r_1 \leq \cdots \leq r_4$ be the eigenvalues of the Ricci tensor of M^4. Assume that the Ricci curvature of M^4 and the scalar curvature of M^4 satisfy $\mathrm{Ric} \geq \rho > 0$ and $R \leq (6 + 2\sqrt{3})\rho$, respectively (the proof of the other case is similar). Comparing (2.6) and (2.8) with (2.10) we have

$$4\pi^2(2\chi + 3\tau) \geq \int_M \Big(-\frac{R^2}{4} + 3\rho R - 6\rho^2\Big)d\mu \geq 0.$$

From this it follows that $\chi \geq -\frac{3\tau}{2}$, which finishes the proof of the theorem. □

2.5 Proof of Theorem 4

Proof Let M^4 be a compact-oriented Einstein four-dimensional manifold with negative scalar curvature R and let $\lambda_+ \leq \mu_+ \leq \nu_+$ be the eigenvalues of W^+ such that $\mu_+ \geq -R/12$. Note that $\lambda_+ + \mu_+ + \nu_+ = 0$ and

$$\mu_+ \geq -R/12 \Rightarrow -\lambda_+ = \mu_+ + \nu_+ \geq 2\mu_+ \geq -R/6 \Rightarrow \lambda_+ \leq R/6 < R/12. \quad (2.11)$$

Since λ_+, μ_+ and ν_+ are continuous eigenvalues of the symmetric operator W^+, if $\lambda_+ < 0$ and $\nu_+ \geq \mu_+ > 0$, we have $\lambda_+ \neq \mu_+$ and $\lambda_+ \neq \nu_+$. Therefore, λ_+ is an eigenvalue of W^+ with constant multiplicity on M^4 and so λ_+ is differentiable on M^4. As W^+ is a harmonic tensor, then (see equation (35) in [7]) there exists an open set A in M^4, such that for all $x \in A$ there exist 1-forms a, b and c defined near of x such that

$$\frac{\Delta\lambda_+}{2} = \lambda_+^2 + 2\mu_+\nu_+ - \frac{R}{4}\lambda_+ + (\nu_+ - \lambda_+) \mid b \mid^2 + (\mu_+ - \lambda_+) \mid c \mid^2 . \quad (2.12)$$

Therefore, from (2.12) we deduce

$$\frac{\Delta\lambda_+}{2} \geq \lambda_+^2 + 2\mu_+\nu_+ - \frac{R}{4}\lambda_+$$

holds on A, thus we obtain

$$0 = \int_M \frac{\Delta\lambda_+}{2} \mathrm{d}\mu \geq \int_M \Big(\lambda_+^2 + 2\mu_+\nu_+ - \frac{R}{4}\lambda_+\Big)\mathrm{d}\mu. \quad (2.13)$$

On the other hand, we have

$$0 < \mu_+ \leq \nu_+ \text{ and } \mu_+ \geq \frac{-R}{12},$$

which implies

$$2\mu_+\nu_+ \geq 2\mu_+^2 \geq \frac{R^2}{72}.$$

From what it follows that

$$\lambda_+^2 + 2\mu_+\nu_+ - \frac{R}{4}\lambda_+ \geq \lambda_+^2 - \frac{R}{4}\lambda_+ + \frac{R^2}{72}.$$

Now, we may use (2.11) to obtain

$$\lambda_+^2 - \frac{R}{4}\lambda_+ + \frac{R^2}{72} = (\lambda_+ - R/6)(\lambda_+ - R/12) \geq 0.$$

Therefore, (2.13) becomes an equality, which implies $\lambda_+ = R/6$, $\mu_+ = \nu_+ = -R/12$ and $|W^+| \neq 0$ on M^4. Now, we may invoke Proposition 5 of [7] to deduce that M^4 or the twofold cover of M^4 admits a Kaehler metric, which finishes the proof of the theorem. □

2.6 Proof of Theorem 5

Proof Since W^+ is harmonic, we recall the Weitzenbock formula (see 16.17 in [4]) to deduce

$$\Delta|W^+|^2 = -R|W^+|^2 + 36 \det W^+ - 2|\nabla W^+|^2, \quad (2.14)$$

where R is the scalar curvature of M^4 and $\det W^+$ is the product of the eigenvalues of W^+.

Taking into account that $|W^+|$ is constant, we deduce from (2.14) that

$$0 \leq -|W^+|^2 \int_M R\mathrm{d}\mu + 36 \int_M \det W^+ \mathrm{d}\mu. \tag{2.15}$$

By use of Lagrange multipliers we have

$$\det W^+ \leq \frac{\sqrt{6}}{18}|W^+|^3. \tag{2.16}$$

Moreover, the equality is attained at points where $W^+ \neq 0$ if and only if W^+ has precisely two different eigenvalues at each point. On the other hand, the trace of the fundamental almost Ricci soliton equation implies

$$\int_M R\mathrm{d}\mu = 4 \int_M \lambda \mathrm{d}\mu,$$

therefore, comparing the previous equation with (2.15) we obtain

$$0 \leq -|W^+|^2 \int_M (-4\lambda + 2\sqrt{6}|W^+|)\mathrm{d}\mu. \tag{2.17}$$

Our assumption in the previous inequality implies that either $|W^+| = 0$ or $|W^+| = \frac{\sqrt{6}}{3V} \int_M \lambda \mathrm{d}\mu$. In the last case, we can use Proposition 5 of [7] to conclude that M^4 is Kaehler–Einstein, which finishes the proof of the theorem. □

2.7 Proof of Corollary 2

Proof Indeed, using Perelman's theorem [14], we may assume that the Ricci soliton is gradient. Moreover, since $|W^+|$ is constant, up to rescaling the metric, we may assume that $\lambda = \frac{\sqrt{6}}{2}|W^+|$. On the other hand, since the self-dual part of the Weyl tensor is harmonic, it is well known that the self-dual part of the Weyl tensor obtained after rescaling the metric will be harmonic. Therefore, the proof of the corollary follows from Theorem 5. □

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