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Involutive constrained systems and Hamilton-Jacobi formalism

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In this paper, we study singular systems with complete sets of involutive constraints. The aim is to establish, within the Hamilton-Jacobi theory, the relationship between the Frobenius' theorem, the infinitesimal canonical transformations generated by constraints in involution with the Poisson brackets, and the lagrangian point (gauge) transformations of physical systems. © 2014 AIP Publishing LLC. [<http://dx.doi.org/10.1063/1.4900921>]

I. INTRODUCTION

Symmetries become the cornerstone of the modern physics, from the advent of the special theory of relativity to the standard model of the elementary particles. Among the physical requirements for the construction of fundamental interactions in quantum field theory is the concept of gauge symmetry and, for the gravitational field, the concept of diffeomorphism invariance in a Riemannian space-time. In the mathematical sense, integrability of ordinary (ODEs) and partial differential equations (PDEs) is usually related to the existence of certain integrals of motion of dynamical systems, and when the physical systems to which those equations belong can be described by variational problems, symmetries are found to be related to the existence of these integrals of motion. The astounding beauty of the subject is that mathematical symmetries determine the physical interaction of the matter fields in nature.

The study of symmetries in field theory is historically linked to the lagrangian and hamiltonian formalisms. In the lagrangian picture we have the Noether's theorems, which relate symmetries of a fundamental integral to conserved currents and geometrical identities in the context of the calculus of variations. The hamiltonian formalism, on the other hand, deals with symmetries in the context of canonical transformations. Even in classical mechanics, where both formalisms are completely equivalent, there is no general and unique correspondence between canonical and lagrangian symmetries.

In field theories, the existence of lagrangian symmetries implies singularity of the lagrangian function, which means that there are constraints relating some of the phase-space variables.^{1,2} In this case, the hamiltonian picture must be constructed to be equivalent to the lagrangian one. The first and most used method to build this equivalence is Dirac's hamiltonian method,³ which consists in the construction of the hamiltonian picture by consistency. For detailed textbooks on the subject, we refer to Ref. 4.

Dirac found that symmetries of a singular system are related to the existence of a subset of the so called primary first-class constraints, which are generators of "gauge transformations" in the hamiltonian method. However, this relationship has no general rule in sight of the applications, and

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he found himself obligated to conjecture that the complete generator of the lagrangian symmetries of a system must be a linear combination of all first-class constraints.⁵ In the mathematical point of view, this problem is still considered unsolved, despite some results in the literature,⁶ and the success in applying the conjecture in the analysis of gauge theories.⁷

On the other hand, the Hamilton-Jacobi (HJ) formalism provides a very natural way of dealing with symmetries, and also with singular systems. Carathéodory⁸ was the first to notice that this formalism is the unifying theory between the theory of first-order PDEs, the theory of first-order ODEs, and the calculus of variations. Lagrangian and hamiltonian dynamics find place as direct mathematical consequences of the HJ theory. With the suggestive name of “the complete figure” of the variational calculus, the HJ formalism was extended to treat singular systems by Güler,⁹ followed by generalizations for higher order derivative lagrangians,¹⁰ Berezin systems,¹¹ linear actions,¹² and applications, specially in the gravitational field¹³ and topologically massive theories.¹⁴ In the HJ formalism, canonical constraints form a set of PDEs of the first-order, and the dynamical evolution is generated by a complete set of independent hamiltonian functions, resulting in a system with several independent variables. In general, a system presents two different sets of constraints, which are called non-involutive and involutive constraints. In Ref. 15, systems with non-involutive constraints are studied, and it is shown that these constraints are responsible to change the symplectic structure of the phase space.

In this paper, we provide a continuation of the work,¹⁵ and study systems with involutive constraints in sight of the Frobenius’ integrability theorem. In Sec. II, we make a review of the HJ formalism. In Sec. III, we present an analytic derivation of the Frobenius’ integrability conditions. In Sec. IV, we turn to a geometrical description of the HJ formalism to show that involutive constraints are generators of canonical transformations on a complete phase space, also discussing the relationship between these transformations and the lagrangian transformations in singular theories. In Sec. V, we present three applications: the first two are mechanical models for gauge theories, the first one having just involutive constraints, the second one presenting involutive and non-involutive ones. The last example is the free Yang-Mills field.

II. THE HAMILTON-JACOBI FORMALISM

In this section, we make a brief review of the HJ formalism for singular systems. Let us consider a system described by the action

$$I = \int_{t_0}^{t_1} dt L(t, q^i, \dot{q}^i), \quad (1)$$

where q^i are N generalized coordinates of a configuration manifold \mathbb{Q}_N , and \dot{q}^i are their respective velocities. Solutions of a variational problem involving (1) are trajectories $q^i = q^i(t)$ parametrized by the time t . Although we are dealing with a classical mechanical approach, extension to field theories is straightforward.

According to Carathéodory,⁸ the necessary condition for the existence of an extreme configuration of (1) is the existence of a function $S(t, q^i)$ that obeys

$$\frac{\partial L}{\partial \dot{q}^i} = \frac{\partial S}{\partial q^i}, \quad (2a)$$

$$\frac{\partial S}{\partial t} + \frac{\partial S}{\partial q^i} \dot{q}^i - L = 0. \quad (2b)$$

The HJ formalism emerges by making (2b) a PDE for S . This can be accomplished if we are able to find expressions of the velocities \dot{q}^i in terms of the coordinates and derivatives of S . Equation (2a) can be inverted to give such expressions if the Hessian condition

$$\det W_{ij} = \det \left(\frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} \right) \neq 0 \quad (3)$$

is satisfied.

However, let us suppose that the Hessian has rank $P \leq N$. This implies a split of \mathbb{Q}_N in two subspaces: the first, \mathbb{Q}_P , is spanned by the P coordinates q^a related to the regular part of the Hessian matrix. The second, Γ_R , is spanned by $R = N - P$ variables $t^z \equiv q^z$ related to the null space of W_{ij} . Then, we are allowed to invert the equations for \dot{q}^a that give us P velocities

$$\dot{q}^a = \phi^a \left(t, t^z, q^b, \frac{\partial S}{\partial q^b} \right), \quad z = 1, \dots, R; \quad a, b = 1, \dots, P. \quad (4)$$

The remaining equations cannot be inverted, but they must be valid nevertheless. We may write them as

$$\frac{\partial S}{\partial t^z} + H_z \left(t, t^z, q^a, \frac{\partial S}{\partial q^a} \right) = 0, \quad H_z \equiv - \left. \frac{\partial L}{\partial \dot{t}^z} \right|_{\dot{q}=\phi}. \quad (5)$$

We suppose that the functions H_z do not depend on \dot{t}^z , because otherwise, the relations (5) would be invertible. We could allow dependence on \dot{t}^z in a non-invertible way, but this dependence would come from very strange lagrangian functions, which we will not consider here.

Using (4) in (2b) we can show that, when (5) are obeyed, the hamiltonian function

$$H_0 \equiv \frac{\partial S}{\partial t^z} \dot{t}^z + \frac{\partial S}{\partial q^a} \phi^a - L(t, t^z, q^a, \dot{t}^z, \phi^a) \quad (6)$$

does not depend on \dot{t}^z . Then, (2b) becomes the desired PDE

$$\frac{\partial S}{\partial t} + H_0 \left(t, t^z, q^a, \frac{\partial S}{\partial q^a} \right) = 0, \quad (7)$$

known as the Hamilton-Jacobi equation.

Along with (7), (5) are also valid, and together they form a set of PDEs for S . Let us define $t^0 \equiv t$, then we are able to write these equations in a unified way:

$$\frac{\partial S}{\partial t^\alpha} + H_\alpha \left(t^\beta, q^a, \frac{\partial S}{\partial q^a} \right) = 0, \quad \alpha, \beta = 0, 1, \dots, R. \quad (8)$$

These are the Hamilton-Jacobi partial differential equations (HJPDEs).

A. The canonical description and characteristic equations

In the HJ theory, the conjugate momenta are defined to be in the direction of the gradient of the function S :

$$\pi_\alpha \equiv \partial S / \partial t^\alpha, \quad p_a \equiv \partial S / \partial q^a. \quad (9)$$

Now, we define the functions

$$H'_\alpha(t^\beta, q^a, \pi_\beta, p_a) \equiv \pi_\alpha + H_\alpha(t^\beta, q^a, p_a). \quad (10)$$

In this case, Eq. (10) is identified with a set of canonical constraints $H'_\alpha = 0$. Therefore, the system should be completely described by the set of HJPDEs

$$H'_\alpha(t^\beta, q^a, \pi_\beta, p_a) = 0, \quad (11a)$$

$$\pi_\alpha = \frac{\partial S}{\partial t^\alpha}, \quad p_a = \frac{\partial S}{\partial q^a}. \quad (11b)$$

The HJ equations (11a) form a set of $R + 1$ PDEs of the first-order. If t^α are independent among each other, we may find a related set of total differential equations (TDEs)

$$dq^a = \frac{\partial H'_\alpha}{\partial p_a} dt^\alpha, \quad dp_a = - \frac{\partial H'_\alpha}{\partial q^a} dt^\alpha, \quad (12a)$$

$$dS = p_a dq^a + \pi_\alpha dt^\alpha - H'_\alpha dt^\alpha. \quad (12b)$$

These are the characteristic equations (CEs) of the HJPDEs. Independence between t^α is assured by an integrability theorem, which will be discussed in Sec. III, but it is important to remark that full integrability of the HJ equations is a necessary condition for the derivation of (12).

The CE has the form of canonical equations with several independent variables t^α as evolution parameters. Complete solutions are congruences of $(R + 1)$ -parameter curves

$$q^a = q^a(t^\alpha), \quad p_a = p_a(t^\alpha) \quad (13)$$

of a reduced phase space $\mathbf{T}^*\mathbb{Q}_P$ spanned by the variables q^a and the conjugate momenta p_a . Observing (13), we call the set $\xi^A \equiv (q^a, p_a)$ the dependent variables of the theory, and the set t^α the independent variables, or parameters. Therefore, it is possible to describe the dynamical evolution of any function $F(t^\alpha, \pi_\alpha, q^a, p_a)$ in an extended phase space $\mathbf{T}^*\mathbb{Q}_{N+1}$ spanned by the complete set of variables $\xi^I \equiv (t^\alpha, \pi_\alpha, q^a, p_a)$. This is achieved by the fundamental differential

$$dF = \{F, H'_\alpha\} dt^\alpha, \quad (14)$$

with the extended Poisson brackets (PBs)

$$\{A, B\} \equiv \frac{\partial A}{\partial t^\alpha} \frac{\partial B}{\partial \pi_\alpha} - \frac{\partial B}{\partial t^\alpha} \frac{\partial A}{\partial \pi_\alpha} + \frac{\partial A}{\partial q^a} \frac{\partial B}{\partial p_a} - \frac{\partial B}{\partial q^a} \frac{\partial A}{\partial p_a}. \quad (15)$$

The functions H'_α are the very generators of the dynamical evolution (14), acting as hamiltonian functions. Therefore, the HJ formalism describes singular systems as several independent variable systems.

III. INTEGRABILITY

The HJ equations (12) become the necessary conditions for the existence of extreme configurations of the action (1), but they are still not sufficient. In deriving the CE we used the fact that the independent variables of the system must be mutually independent. However, this cannot be generally assured only by the HJ equations. Independence of the parameters is related to the fact that the evolution of the system in the direction of an independent variable should be independent of the other variables. On the other hand, this is related to the very integrability of the theory, which means the existence of complete solutions of the HJPDEs, as well as the existence of a unique solution of the CE once given a set of initial conditions. In this section, we discuss what are the conditions that the HJ equations must obey to be a complete integrable system of PDEs. This can be accomplished in several ways. In our discussion, we generalize the method presented in Ref. 16.

A. The Lagrange brackets

Let us suppose the set of HJ equations (11a) to be satisfied. If they form a complete integrable set, there exists a complete solution with the form

$$S = S[t^\alpha, q^a(t^\alpha)]. \quad (16)$$

The function S is submitted to the conditions (11b), and the functions $q^a = q^a(t^\alpha)$, $p_a = p_a(t^\alpha)$ are supposed to be solutions of the CE (12a).

We may take the derivative

$$\frac{dS}{dt^\alpha} = \frac{\partial S}{\partial t^\alpha} + \frac{\partial S}{\partial q^a} \frac{dq^a}{dt^\alpha} = \pi_\alpha + p_a \frac{dq^a}{dt^\alpha} \equiv p_{i'} \frac{dq^{i'}}{dt^\alpha}, \quad \{i'\} = \{0, 1, \dots, N\}.$$

The second derivative results

$$\frac{d^2 S}{dt^\alpha dt^\beta} - p_{i'} \frac{d^2 q^{i'}}{dt^\alpha dt^\beta} = \frac{dq^{i'}}{dt^\alpha} \frac{dp_{i'}}{dt^\beta}.$$

The left-hand side is symmetric in α and β , so the skew-symmetric part of the right-hand side must be zero. This yields the condition

$$(t^\alpha, t^\beta) = 0, \quad (17)$$

where we define the Lagrange brackets on the complete phase space:

$$(t^\alpha, t^\beta) \equiv \frac{dq^{i'} dp_{i'}}{dt^\alpha dt^\beta} - \frac{dq^{i'} dp_{i'}}{dt^\beta dt^\alpha}. \tag{18}$$

Therefore, the condition (17) is necessary for the existence of a complete solution of the HJ equations.

To show that (17) is also sufficient, let us suppose a set of functions

$$q^{i'} = q^{i'}(t^\alpha), \quad p_{i'} = p_{i'}(t^\alpha)$$

that obeys (17). Taking total derivatives of (17), we have

$$\frac{d}{dt^\beta} \left[\frac{dq^{i'}}{dt^\alpha} p_{i'} \right] = \frac{d}{dt^\alpha} \left[\frac{dq^{i'}}{dt^\beta} p_{i'} \right].$$

Observing the above expression, there must be a function $S [q^{i'}(t^\alpha)]$ such that

$$\frac{dq^{i'}}{dt^\alpha} p_{i'} = \frac{dS}{dt^\alpha}. \tag{19}$$

In this case, derivation of S yields

$$\frac{dS}{dt^\alpha} = \frac{\partial S}{\partial q^{i'}} \frac{dq^{i'}}{dt^\alpha}, \tag{20}$$

and comparing (19) and (20),

$$p_{i'} = \frac{\partial S}{\partial q^{i'}} \Rightarrow p_a = \frac{\partial S}{\partial q^a}, \quad \pi_\alpha = \frac{\partial S}{\partial t^\alpha}. \tag{21}$$

Therefore, (17) is the necessary and sufficient condition for the existence of a function $S(t^\alpha, q^a)$ whose gradient follows the direction of the conjugate momenta.

If we take the derivative

$$\frac{d}{dt^\alpha} \left[p_{i'} - \frac{\partial S}{\partial q^{i'}} \right] = \frac{dp_{i'}}{dt^\alpha} - \frac{\partial^2 S}{\partial q^{i'} \partial t^\alpha} - \frac{\partial^2 S}{\partial q^{i'} \partial q^{j'}} \frac{dq^{j'}}{dt^\alpha} = 0,$$

and use (12a), we see that

$$\frac{dH'_\alpha}{dq^{i'}} = 0.$$

The general solution is given by

$$H'_\alpha(t^\beta, q^a, \pi_\beta, p_a) = \text{constant}, \tag{22}$$

where the constant can be taken to be zero without loss of generality.

Therefore, (16) is a solution of the HJPDEs, and the conditions (17) are the necessary and sufficient conditions for the existence of a complete solution of these equations. They become our first version of the integrability conditions.

B. Frobenius' integrability conditions

The conditions (17) are not very useful, since they demand knowledge of the solutions of the variational problem. However, using the CEs (12a) we may show that

$$(t^\alpha, t^\beta) = \frac{\partial H'_\alpha}{\partial t^\gamma} \frac{\partial H'_\beta}{\partial \pi_\gamma} - \frac{\partial H'_\beta}{\partial t^\gamma} \frac{\partial H'_\alpha}{\partial \pi_\gamma} + \frac{\partial H'_\alpha}{\partial q^a} \frac{\partial H'_\beta}{\partial p_a} - \frac{\partial H'_\beta}{\partial q^a} \frac{\partial H'_\alpha}{\partial p_a} = \{H'_\alpha, H'_\beta\}.$$

Therefore, the integrability conditions (17) can be written as

$$\{H'_\alpha, H'_\beta\} = 0. \tag{23}$$

The conditions on the Lagrange brackets of the independent variables becomes conditions on the Poisson brackets of the generators. Equation (23) is known as the Frobenius' integrability conditions (FICs).

Note that

$$dH'_\alpha = \{H'_\alpha, H'_\beta\} dt^\beta,$$

and, if t^α are independent parameters, the FICs imply

$$dH'_\alpha = 0. \quad (24)$$

The conditions (24) and (23) are completely equivalent, but the latter is more convenient to analyze systems that are not integrable at first sight. As shown in Ref. 15, application of (24) may reveal dependence of the independent variables, in the form of total differential equations, leading naturally to the introduction of generalized brackets. Moreover, (24) states that the generators H'_α are also a set of dynamical invariants.

We may generalize (23) to

$$\{H'_\alpha, H'_\beta\} = C_{\alpha\beta}{}^\gamma H'_\gamma. \quad (25)$$

The proof for the case of classical mechanics can be found in Ref. 17, but involves a highly mathematical labor. However, we can easily convince ourselves that (25) is a proper generalization. First, we notice that the meaning of (24) is that the generators must be dynamical invariants. Second, we may use the Jacobi identity to show that the PBs of two dynamical invariants are another dynamical invariant. If the set H'_α closes the Poisson algebra (25), it means that this set is a complete set of invariants. In this case, we clearly have preserved the relations (24) in the reduced phase space, where $H'_\alpha = 0$, since (25) implies

$$dH'_\alpha = \{H'_\alpha, H'_\beta\} dt^\beta = C_{\alpha\beta}{}^\gamma H'_\gamma dt^\beta = 0.$$

Therefore, even if (25) are obeyed instead of the stronger conditions (23), (24) still hold in $\mathbf{T}^*\mathbb{Q}_P$, and the dynamics in this reduced phase space is independent for each parameter t^α .

IV. CANONICAL TRANSFORMATIONS GENERATED BY A COMPLETE SET OF INVOLUTIVE CONSTRAINTS

In regular classical mechanics, temporal evolution can be seen as a set of successive infinitesimal canonical transformations.¹⁸ This becomes evident since we may write solutions of Hamilton's equations as hamiltonian flows generated by the hamiltonian function, and these flows are canonical in the sense that they preserve the volume of the phase space (Liouville theorem). For constrained systems we may show that this picture is also valid. In order to build this picture we now turn to the geometrical aspects of the HJ formalism.

A. The geometric approach

Let us define the set of vector fields

$$X_\alpha \equiv \frac{d}{dt^\alpha} = \chi_\alpha^I \frac{\partial}{\partial \xi^I}, \quad \chi_\alpha^I \equiv \{ \xi^I, H'_\alpha \}, \quad (26)$$

where $\xi^I = (t^\alpha, q^a, \pi_\alpha, p_a)$. With these vector fields, the fundamental differential (14) can be written as

$$dF = dt^\alpha X_\alpha F. \quad (27)$$

We may also rewrite the CEs using (27). They have the form of the TDEs

$$d\xi^I = dt^\alpha X_\alpha \xi^I, \quad dS = dt^\alpha X_\alpha S. \quad (28)$$

Now we compute the Lie derivative between two of these vector fields:

$$\mathcal{L}_{X^\alpha} X_\beta F = [X_\alpha, X_\beta] F = \{ \{ F, H'_\beta \}, H'_\alpha \} - \{ \{ F, H'_\alpha \}, H'_\beta \}.$$

Applying the Jacobi identity on the right-hand side, also considering (25), we have

$$\mathcal{L}_{X^\alpha} X_\beta F = -C_{\alpha\beta}{}^\gamma X_\gamma F + \{C_{\alpha\beta}{}^\gamma, F\} H'_\gamma.$$

If the structure coefficients are independent of ξ^I , the integrability conditions become conditions over the Lie brackets between the vector fields,

$$[X_\alpha, X_\beta] = f_{\alpha\beta}{}^\gamma X_\gamma, \quad f_{\alpha\beta}{}^\gamma \equiv -C_{\alpha\beta}{}^\gamma. \quad (29)$$

These are also sufficient conditions for a set of vectors X_α to be a complete basis, therefore these vectors span a vector space of dimension $R + 1$. We may identify this vector space with Γ_{R+1} , which is the space of the independent variables t^α .

B. The symplectic structure

We may build the symplectic structure considering the Pfaffian 1-form defined by $\theta_c \equiv dS$. According to the CE (12b),

$$\theta_c = p_a dq^a + \pi_\alpha dt^\alpha - H'_\alpha dt^\alpha. \quad (30)$$

The symplectic 2-form is defined by $\omega \equiv -d\theta_c$, which gives the expression

$$\omega = \omega_P + a, \quad (31)$$

where

$$\omega_P \equiv dq^a \wedge dp_a, \quad (32a)$$

$$a \equiv dt^\alpha \wedge d\pi_\alpha + dH'_\alpha \wedge dt^\alpha. \quad (32b)$$

The 2-form ω_P can be written by

$$\omega_P = dq^a \wedge dp_a = \frac{1}{2} d\xi^A \omega_{AB} d\xi^B, \quad (33)$$

where $\xi^A = (q^a, p_a)$. The matrix ω_{AB} is given by

$$\omega_{AB} \equiv \begin{pmatrix} 0 & \delta_{ab} \\ -\delta_{ab} & 0 \end{pmatrix}. \quad (34)$$

On the other hand, since $dH'_\alpha = 0$,

$$a = dt^\alpha \wedge d\pi_\alpha = \left(\frac{\partial^2 S}{\partial t^\alpha \partial t^\beta} \right) dt^\alpha \wedge dt^\beta. \quad (35)$$

The expression in brackets is symmetric in α and β , therefore a is identically zero.

We see that the symplectic structure ω is singular. Under the assumption of integrability, it becomes the sum of a regular 2-form ω_P and a null 2-form a . We notice that ω_P is non-degenerate. This is not the case of the full 2-form ω : if we take the vectors X_α we get the contraction

$$dH'_\alpha = i_{X_\alpha} \omega = 0, \quad (36)$$

if (24) hold. Therefore, we recover the geometric nature of the singularity of a given system: it comes from the fact that the symplectic structure is degenerate, and the vector fields X_α are the eigenvectors that correspond to its null space. We also have

$$i_{X_\alpha} i_{X_\beta} \omega = \{H'_\alpha, H'_\beta\}, \quad (37)$$

so the FICs (23) can also be written as $i_{X_\alpha} i_{X_\beta} \omega = 0$.

C. Canonical transformations and characteristic flows

Now let us see how the vector fields X_α generate active canonical transformations in $\mathbf{T}^*\mathbb{Q}_{N+1}$. The general form of these transformations is naturally given by the structure of the fundamental differential (27). Let us define an infinitesimal transformation $\delta t^\alpha \equiv \bar{t}^\alpha - t^\alpha$ on the independent variables. In principle, δt^α are arbitrary (small) functions of ξ^I . In this case, it implies the transformation

$$\delta F = \delta t^\alpha X_\alpha F, \tag{38}$$

for any function $F(\xi^I)$ of $\mathbf{T}^*\mathbb{Q}_{N+1}$. We remark that this transformation is not generally related to the dynamics, so the characteristic equations and the HJ equations $H'_\alpha = 0$ may not be satisfied. However, if we choose $\delta t^\alpha = dt^\alpha$, (38) becomes the fundamental differential (27) of the system. Therefore, the dynamical evolution becomes a special case of the transformation (38).

If $F = \xi^I$, (38) defines transformations in the coordinates of $\mathbf{T}^*\mathbb{Q}_{N+1}$. We may write these transformations as

$$\xi^I(\bar{t}^\alpha) = g \xi^I(t^\alpha), \tag{39}$$

where we define the operator

$$g \equiv 1 + \delta t^\alpha X_\alpha. \tag{40}$$

In this case we say that g carries the infinitesimal flows generated by X_α . Let us call these flows the characteristic flows (CFs) of the system.

The CFs are active canonical transformations. We may see this by taking the application $g\omega g^{-1}$, where $g^{-1} \equiv 1 - \delta t^\alpha X_\alpha$ is the inverse transformation. Supposing integrability, we have

$$g g^{-1} = g^{-1} g = 1, \tag{41}$$

and then,

$$g\omega g^{-1} = \omega - \delta t^\alpha \delta t^\beta i_{X_\alpha} i_{X_\beta} \omega = \omega. \tag{42}$$

So ω is preserved by (38). Of course, invariance of the symplectic 2-form ω is reflected in any $2p$ -form

$$\omega^{\wedge p} \equiv \underbrace{\omega \wedge \omega \wedge \dots \wedge \omega}_p,$$

specially the volume $2(N + 1)$ -form $v \approx \omega^{\wedge 2(N+1)}$, whose invariance is known as the Liouville theorem. Because $\omega = \omega_p + a$, and a is a null-form, the volume element $a^{\wedge 2(R + 1)}$ is identically zero, and all the above properties are also applied to ω_p .

Now, suppose a 2-dimensional surface $\Lambda \subset \mathbf{T}^*\mathbb{Q}_{N+1}$. The area of this surface is calculated by

$$I_\Lambda = \int_\Lambda \omega = - \int_\Lambda d\theta_c = - \oint_{\partial\Lambda} \theta_c.$$

Then, invariance of ω implies that I_Λ is preserved by the CFs. On the right side we have the integral of the canonical 1-form θ_c over a closed curve $\partial\Lambda$. If this integral is preserved, the integral

$$S = \int_C \theta_c \tag{43}$$

is path independent. This integral defines a canonical fundamental integral in $\mathbf{T}^*\mathbb{Q}_p$, canonical action that was already found in the form of the total differential equation (12b). This action is, then, invariant under the transformations (38) apart of boundary terms.

Let us suppose the case in which the FICs (25) imply

$$[X_\alpha, X_\beta] = f_{\alpha\beta}{}^\gamma X_\gamma, \tag{44}$$

which happens to be the necessary and sufficient conditions for X_α to be a complete basis of $\Gamma_{R + 1}$. Supposing a function $F \in \mathbf{T}^*\mathbb{Q}_{N+1}$, we may build the composition of two flows g_ϵ and g_λ

$$g_\epsilon \equiv 1 + \epsilon^\alpha X_\alpha, \quad g_\lambda \equiv 1 + \lambda^\alpha X_\alpha.$$

This composition yields the Lie bracket

$$[g_\epsilon, g_\lambda] F = g_\nu F, \quad g_\nu \equiv 1 + \nu^\alpha X_\alpha, \quad \nu^\gamma \equiv \epsilon^\alpha \lambda^\beta f_{\alpha\beta}{}^\gamma.$$

Therefore, the composition of two characteristic flows is another characteristic flow. This is the group property that allows the finite composition

$$g(\Delta t^\alpha) = \exp[\Delta t^\alpha X_\alpha], \quad (45)$$

to become an element of a Lie group of canonical transformations. In other words, if the algebra of the involutive constraints is reflected on the vector fields X_α , we may build a Lie group out of the Lie algebra of those vector fields.

As result, we reach our first objective, which is to show that a complete set of involutive constraints $H'_\alpha = 0$ are generators of infinitesimal canonical transformations with the form

$$\delta\xi^I = \{\xi^I, H'_\alpha\} \delta t^\alpha, \quad (46)$$

which are called the characteristic flows of the system.

D. Connection to gauge transformations

Among the CFs (46), we may define a special class of transformations. We set $\delta t^0 = \delta t = 0$, which means that these transformations are taken at constant time. They are given by

$$\delta\xi^I = \{\xi^I, H'_z\} \delta t^z. \quad (47)$$

Although (47) has the same structure of the canonical flows, this is not sufficient to assure canonicity. The generators must be in involution among themselves,

$$\{H'_z, H'_x\} = C_{zx}{}^y H'_y. \quad (48)$$

Remember that the algebra that assures integrability is the complete Poisson algebra given by (25), which includes the generator of time displacement H'_0 . The components (z, x) of that expression are given by

$$\{H'_z, H'_x\} = C_{zx}{}^y H'_y + C_{zx}{}^0 H'_0,$$

so (48) does not hold unless the structure coefficients $C_{zx}{}^0$ are zero, or the constraints $H'_\alpha = 0$ are valid. If we impose $C_{zx}{}^0 = 0$, it is implied that the bracket $\{H'_z, H'_0\}$ must be identically zero in the complete phase space $\mathbf{T}^*\mathbb{Q}_{N+1}$. This restriction is simply too strong and cannot be accomplished in general. On the other hand, we may demand that the HJ equations $H'_\alpha = 0$ are valid. In this case, the algebrae (25) and (48) become Abelian, and the transformations (47) become restricted to $\mathbf{T}^*\mathbb{Q}_P$.

Therefore, along with the condition that $\delta\xi^I$ are taken at constant t , we may impose that they cannot leave the reduced phase space. In this case, (47) become the same transformations that were called “point transformations” by Dirac,⁵ in the hamiltonian picture.

The point transformations (47) are generated by the function

$$G = H'_z \delta t^z, \quad (49)$$

since

$$\delta_G \xi^I = \{\xi^I, G\} = \{\xi^I, H'_z\} \delta t^z, \quad (50)$$

when $H'_z = 0$. Therefore, $\delta_G \xi^I$ becomes equal to (47) in $\mathbf{T}^*\mathbb{Q}_P$, and G is called their generating function. These transformations are canonical symmetries, since the algebra (48) assures that the symplectic structure is not changed by (50).

On the other hand, gauge transformations are Noether symmetries of the fundamental integral. Suppose a set of infinitesimal transformations $\delta t = \bar{t} - t$ and $\delta q^i = \bar{q}^i - q^i$, the change in the action (1) is given by

$$\delta I = \int dt \left[\left(\frac{\partial L}{\partial q^i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} \right) \left(\delta q^i - \delta t \frac{dq^i}{dt} \right) + \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^i} \delta q^i - H \delta t \right) \right], \quad (51)$$

where

$$H \equiv \frac{\partial L}{\partial \dot{q}^i} \dot{q}^i - L \quad (52)$$

is the hamiltonian function in terms of coordinates and velocities. Equation (51) provides the equations of motion

$$\frac{\partial L}{\partial q^i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} = 0, \quad (53)$$

and the conjugated momenta $p_i = \partial L / \partial \dot{q}^i$ of the system, when Hamilton's principle is applied.

To be lagrangian symmetries, however, the transformations (50), with $\delta t = 0$, must obey the Lie equation¹⁹

$$\delta L = \left(\frac{\partial L}{\partial q^i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} \right) \delta q^i + \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^i} \delta q^i \right) = 0, \quad (54)$$

for the coordinate transformations $\delta q^i = \{q^i, H_z\} \delta t^z = \chi_z^i \delta t^z$. In the following applications, we will see that (54) implies linear dependency between some of the variations δt^z . The generator of the CFs (49), then, becomes the generator of the point transformations, usually up to a total time derivative. For field theories, (51), (52), and (54) have straightforward generalizations.

V. APPLICATIONS

In this section, we apply the ideas discussed in Secs. II–IV in specific examples. The main script is the following: first we find the complete set of involutive constraints of the theory, therefore building its integrability. If necessary, non-involutive constraints are treated by the method developed in Ref. 15, and all Poisson brackets must be changed to generalized brackets. We then build the CEs of the theory, and proceed to the analysis of their characteristic flows and related gauge transformations.

A. The Christ-Lee model

We begin with the Christ-Lee model.²⁰ It can be considered a toy model in classical mechanics, but unlike the usual toy models in the literature, the Christ-Lee system is a very conceivable mechanical system, although a very special one. It is simply a particle on a plane, whose position is given by a vector $\mathbf{x} = (x_1, x_2)$, submitted to the constraint that its position and momentum lie in the same direction.

The lagrangian of the Christ-Lee model is given by

$$L(\mathbf{x}, \dot{\mathbf{x}}, q) = \frac{1}{2} \dot{\mathbf{x}}^2 - q \mathbf{x} \cdot \boldsymbol{\varepsilon} \cdot \dot{\mathbf{x}} + \frac{1}{2} q^2 \mathbf{x}^2 - V(\mathbf{x}^2). \quad (55)$$

The matrix $\boldsymbol{\varepsilon}$ is the skew-symmetric matrix

$$\boldsymbol{\varepsilon} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (56)$$

In this, and in the next example, we use the dot “ \cdot ” to denote a scalar product. In this case $\dot{\mathbf{x}}^2$ represents $x_i \dot{x}^i$, $\mathbf{x} \cdot \boldsymbol{\varepsilon} \cdot \dot{\mathbf{x}}$ actually means $\varepsilon_{ij} x^i \dot{x}^j$, and $\boldsymbol{\varepsilon} \cdot \mathbf{x}$ states for $\varepsilon_{ij} x^j = -\varepsilon_{ji} x^i$, where $i, j = 1, 2$.

The Euler-Lagrange equations of the Christ-Lee model are

$$\mathbf{x} \cdot (q \mathbf{x} - \boldsymbol{\varepsilon} \cdot \dot{\mathbf{x}}) = 0, \quad (57a)$$

$$\ddot{\mathbf{x}} + 2q \boldsymbol{\varepsilon} \cdot \dot{\mathbf{x}} + \dot{q} \boldsymbol{\varepsilon} \cdot \mathbf{x} - q^2 \mathbf{x} = -\nabla V, \quad (57b)$$

and the momenta

$$\mathbf{p} \equiv \dot{\mathbf{x}} - q \mathbf{x} \cdot \boldsymbol{\varepsilon}, \quad \pi = 0, \quad (58)$$

conjugated with the variables \mathbf{x} and q , respectively. The definition for \mathbf{p} gives equations for the velocities

$$\dot{\mathbf{x}} = \mathbf{p} + q \mathbf{x} \cdot \boldsymbol{\varepsilon}, \quad (59)$$

and the last momentum is a canonical constraint

$$H'_1 \equiv \pi = 0. \quad (60)$$

The canonical hamiltonian is given by

$$H_0 = \frac{1}{2} \mathbf{p}^2 + V + q \mathbb{L}, \quad (61)$$

where $\mathbb{L} \equiv \mathbf{x} \cdot \boldsymbol{\varepsilon} \cdot \mathbf{p} = x_1 p_2 - x_2 p_1$ is the angular momentum in two dimensions. In this case we have two HJ equations:

$$H'_0 \equiv \pi_0 + H_0 = 0, \quad (62a)$$

$$H'_1 \equiv \pi = 0. \quad (62b)$$

Using the fundamental PBs $\{\mathbf{x}, \mathbf{p}\} = \mathbf{1}$ and $\{q, \pi\} = 1$, where $\mathbf{1}$ is the identity in two dimensions, we verify that the PB between (62) are $\{H'_1, H'_1\} = 0$, and $\{H'_1, H'_0\} = -\mathbb{L}$. Then the system of HJ equations is not integrable, requiring the imposition of another constraint,

$$H'_2 \equiv \mathbb{L} = 0. \quad (63)$$

It is straightforward to show that the system (62) and (63) is completely integrable. The final algebra is given by

$$\{H'_1, H'_0\} = -H'_2, \quad (64)$$

with all other PBs identically zero.

The fundamental differential of an observable F is given by

$$dF = \{F, H'_\alpha\} dt^\alpha = \{F, H'_0\} dt^0 + \{F, H'_1\} dt^1 + \{F, H'_2\} dt^2, \quad (65)$$

where $t^\alpha = (t^0 = t, t^1 = q, t^2)$ are the independent variables. Because H'_2 comes from the integrability conditions, we must expand the parameter space, and therefore the complete phase space, with a new parameter t^2 . This is done so that H'_2 becomes a generator of the dynamics in the direction of t^2 .

The characteristic equations are given by

$$dq = dt^1, \quad (66a)$$

$$d\pi = -H'_2 dt, \quad (66b)$$

$$d\mathbf{x} = [\mathbf{p} - q \boldsymbol{\varepsilon} \cdot \mathbf{x}] dt + \boldsymbol{\varepsilon} \cdot \mathbf{x} dt^2, \quad (66c)$$

$$d\mathbf{p} = -[q \boldsymbol{\varepsilon} \cdot \mathbf{p} + \nabla V] dt + \boldsymbol{\varepsilon} \cdot \mathbf{p} dt^2. \quad (66d)$$

Since integrability is assured, t^2 and t are linearly independent, then the time evolution gives the set of equations

$$\dot{q} = 0, \quad (67a)$$

$$\dot{\pi} = -H'_2 = 0, \quad (67b)$$

$$\dot{\mathbf{x}} = \mathbf{p} - q \boldsymbol{\varepsilon} \cdot \mathbf{x}, \quad (67c)$$

$$\dot{\mathbf{p}} = -q \boldsymbol{\varepsilon} \cdot \mathbf{p} - \nabla V. \quad (67d)$$

Relation (67c) reproduces (59). Taking the derivative in t , substituting (67d) and using (67c) again, we have the Euler-Lagrange equations (57b). Then, equivalence is actually assured between the time evolution of the CEs and the Euler-Lagrange equations.

Apart of the time evolution, the generators H'_1 and H'_2 give the canonical transformations $\delta F = \{F, H'_z\} \delta t^z$, for $z = 1, 2$. It results in

$$\delta q = \delta t^1, \quad \delta \mathbf{x} = \boldsymbol{\varepsilon} \cdot \mathbf{x} \delta t^2, \quad \delta \mathbf{p} = \boldsymbol{\varepsilon} \cdot \mathbf{p} \delta t^2. \quad (68)$$

The transformation on q is just arbitrary, it depends of the form of δt^1 . On the other hand, the CT on \mathbf{x} and \mathbf{p} are infinitesimal rotations. The generator of the characteristic flows in the directions of t^1 and t^2 is given by

$$G_{CF} = H'_z \delta t^z = H'_1 \delta t^1 + H'_2 \delta t^2. \quad (69)$$

As we saw earlier, the algebra of the generators H'_1 and H'_2 is Abelian.

Now, the first variation of (55) under infinitesimal transformations $\delta \mathbf{x} \equiv \mathbf{x}' - \mathbf{x}$ and $\delta q = q' - q$ is given by

$$\begin{aligned} \delta L = & \frac{d}{dt} [(\dot{\mathbf{x}} - q \mathbf{x} \cdot \boldsymbol{\varepsilon}) \cdot \delta \mathbf{x}] \\ & - \delta \mathbf{x} \cdot [\ddot{\mathbf{x}} + 2q \boldsymbol{\varepsilon} \cdot \dot{\mathbf{x}} + \dot{q} \boldsymbol{\varepsilon} \cdot \mathbf{x} - q^2 \mathbf{x} + \nabla V] \\ & + \delta q \mathbf{x} \cdot (q \mathbf{x} - \boldsymbol{\varepsilon} \cdot \dot{\mathbf{x}}). \end{aligned} \quad (70)$$

If (68) are symmetries of the lagrangian function, δL must be zero. Then, substituting (68) in (70), and considering $V = V(\mathbf{x}^2)$, we have

$$\delta L = \left(\frac{d}{dt} \delta t^2 + \delta t^1 \right) (q \mathbf{x}^2 - \mathbf{x} \cdot \boldsymbol{\varepsilon} \cdot \dot{\mathbf{x}}). \quad (71)$$

In this case $\delta L = 0$ if $\delta t^1 = -d(\delta t^2)/dt$, independently of the Euler-Lagrange equations. Let us suppose that $\delta t^2 = \theta(t)$. In this case, $\delta t^1 = -\dot{\theta}$, and

$$G_g = -H'_1 \dot{\theta} - H'_2 \theta = -(\pi \dot{\theta} + \mathbb{L} \theta) \quad (72)$$

is the generator of the gauge transformations

$$\delta q = \{q, G\} = -\dot{\theta}, \quad (73a)$$

$$\delta \mathbf{x} = \{\mathbf{x}, G\} = -\mathbf{x} \cdot \boldsymbol{\varepsilon} \cdot \{\mathbf{x}, \mathbf{p}\} \theta = \boldsymbol{\varepsilon} \cdot \mathbf{x} \theta. \quad (73b)$$

B. Chern-Simons quantum mechanics

Now let us consider the two dimensional movement of a charged particle in a constant magnetic field B , and a quadratic scalar potential. This system is described by the Lagrange function

$$L(\mathbf{x}, \dot{\mathbf{x}}) = \frac{1}{2} m \dot{\mathbf{x}}^2 + \frac{B}{2} \mathbf{x} \cdot \boldsymbol{\varepsilon} \cdot \dot{\mathbf{x}} - \frac{k}{2} \mathbf{x}^2,$$

and it is known to be the mechanical analogous of the three-dimensional topologically massive electrodynamics in the Weyl gauge. The term of the magnetic field actually corresponds to a pure Chern-Simons term in the three-dimensional gauge theory. In the limit $m, k \rightarrow 0$ the quantization of this model results in a quantum mechanical theory with interesting topological effects.²¹

This model can be made more interesting with the inclusion of another Chern-Simons term, so we will work with the system described by the function

$$L(\mathbf{x}, \dot{\mathbf{x}}, q) = \frac{B}{2} \mathbf{x} \cdot \boldsymbol{\varepsilon} \cdot D\mathbf{x} + \nu q, \quad D\mathbf{x} \equiv \dot{\mathbf{x}} + q \boldsymbol{\varepsilon} \cdot \mathbf{x}. \quad (74)$$

As in the past example, \mathbf{x} is a position vector in two dimensional euclidian space, q is an auxiliary scalar variable, and ν is a numerical parameter. The matrix $\boldsymbol{\varepsilon}$ is the same defined in (56).

The equations of motion of the Chern-Simons quantum mechanics are

$$\nu = \frac{B}{2} \mathbf{x}^2, \quad (75a)$$

$$D\mathbf{x} = 0, \quad (75b)$$

and the conjugate momenta of the variables q and \mathbf{x} ,

$$\pi_q = 0, \quad \mathbf{p} = -\frac{B}{2} \boldsymbol{\varepsilon} \cdot \mathbf{x}, \quad (76)$$

respectively. The canonical hamiltonian takes the form

$$H_0 = -q \left(v - \frac{B}{2} \mathbf{x}^2 \right). \quad (77)$$

Therefore, we have the following set of HJPDEs:

$$H'_0 \equiv \pi_t + H_0 = 0, \quad (78a)$$

$$H'_1 \equiv \pi_q = 0, \quad (78b)$$

$$\boldsymbol{\psi} \equiv \mathbf{p} + \frac{B}{2} \boldsymbol{\varepsilon} \cdot \mathbf{x} = 0. \quad (78c)$$

We notice that the last constraint is a 2-vector.

This system is not integrable: here we have an example where non-involutive constraints are present. This can be seen by the PB

$$\{\boldsymbol{\psi}, \boldsymbol{\psi}\} = B\boldsymbol{\varepsilon}, \quad (79a)$$

$$\{\boldsymbol{\psi}, H'_0\} = qB\mathbf{x}, \quad (79b)$$

$$\{H'_1, H'_0\} = v - \frac{B}{2} \mathbf{x}^2. \quad (79c)$$

Following the procedure outlined in Ref. 15, (79a) indicates that we may introduce the GB

$$\{F, G\}^* = \{F, G\} - \frac{1}{B} \{F, \boldsymbol{\psi}\} \cdot \boldsymbol{\varepsilon} \cdot \{\boldsymbol{\psi}, G\}, \quad (80)$$

which define the fundamental relations

$$\{q, \pi_q\}^* = 1, \quad \{\mathbf{x}, \mathbf{x}\}^* = \frac{1}{B} \boldsymbol{\varepsilon}, \quad \{\mathbf{x}, \mathbf{p}\}^* = \frac{1}{2} \mathbf{1}, \quad \{\mathbf{p}, \mathbf{p}\}^* = -\frac{B}{4} \boldsymbol{\varepsilon}. \quad (81)$$

Because $\{\boldsymbol{\psi}, F\}^* = 0$ identically for any F , all GB between the constraints are zero, except

$$\{H'_1, H'_0\}^* = v - \frac{B}{2} \mathbf{x}^2. \quad (82)$$

Therefore, a new HJ equation

$$H'_2 \equiv v - \frac{B}{2} \mathbf{x}^2 = 0 \quad (83)$$

should be added to the system (78). It is straightforward to show that integrability for H'_3 is obeyed, then the system is completely integrable with the GB (80). Particularly, we have $\{H'_2, H'_1\}^* = 0$.

Equations of motion are calculated by the fundamental differential

$$dF = \{F, H'_\alpha\}^* dt^\alpha = \{F, H'_0\}^* dt^0 + \{F, H'_1\}^* dt^1 + \{F, H'_2\}^* dt^2, \quad (84)$$

where $t^\alpha = (t^0 = t, t^1 = q, t^2)$ are the independent variables. As in the preceding example, the generator H'_2 demands the expansion of the parameter space with the inclusion of a new independent variable t^2 . The characteristic equations are given by

$$dq = dt^1, \quad (85a)$$

$$d\pi = H'_2 dt, \quad (85b)$$

$$d\mathbf{x} = -q\boldsymbol{\varepsilon} \cdot \mathbf{x} dt + \boldsymbol{\varepsilon} \cdot \mathbf{x} dt^2, \quad (85c)$$

$$d\mathbf{p} = \frac{1}{2} B q \mathbf{x} dt - \frac{1}{2} B \mathbf{x} dt^2. \quad (85d)$$

The first equation identifies t^1 with q apart of an arbitrary constant. Equation (85b) reproduces the IC for the involutive constraint H'_1 . This is equivalent to the constraint $H'_2 = 0$, as expected, and therefore Eq. (75a) is achieved.

For the remaining equations, we see that time evolution alone gives

$$\dot{\mathbf{x}} = -q\varepsilon \cdot \mathbf{x}, \quad (86)$$

which is the same as the Euler-Lagrange equation (75b). On the other hand, (85d) becomes

$$\dot{\mathbf{p}} = \frac{1}{2}Bq\mathbf{x}. \quad (87)$$

This equation is the time derivative of $\psi = 0$ when (86) is considered. It actually gives Newton's second law for this system.

In addition to this analysis, we write down the canonical transformations $\delta F = \{F, H'_z\} \delta t^z$, for $z = 1, 2$,

$$\delta q = \delta t^1, \quad \delta \mathbf{x} = \varepsilon \cdot \mathbf{x} \delta t^2, \quad \delta \mathbf{p} = -\frac{1}{2}B\mathbf{x} \delta t^2. \quad (88)$$

The transformation for q is an arbitrary rescaling, and $\delta \mathbf{x}$ is again an infinitesimal rotation. On the other hand, $\delta \mathbf{p}$ is a transformation that mixes positions and momenta of the phase space. It is also straightforward to write the generator of the CFs

$$G_{CF} = H'_z \delta t^z = H'_1 \delta t^1 + H'_2 \delta t^2. \quad (89)$$

Now, the first variation of the lagrangian function (74) under transformations of the form $\delta \mathbf{x} \equiv \mathbf{x}' - \mathbf{x}$ and $\delta q \equiv q' - q$ is given by

$$\delta L = \frac{d}{dt} \left(\frac{B}{2} \mathbf{x} \cdot \varepsilon \cdot \delta \mathbf{x} \right) - B D\mathbf{x} \cdot \varepsilon \cdot \delta \mathbf{x} + \left(v - \frac{B}{2} \mathbf{x}^2 \right) \delta q. \quad (90)$$

For the transformations (88), the variation (90) becomes

$$\begin{aligned} \delta L &= -\frac{B}{2} \mathbf{x}^2 \left(\frac{d}{dt} \delta t^2 + \delta t^1 \right) + v \delta t^1 \\ &= -\frac{B}{2} \mathbf{x}^2 \left[\frac{d}{dt} \delta t^2 + \delta t^1 - \delta t^1 \right] = -\frac{B}{2} \mathbf{x}^2 \frac{d}{dt} \delta t^2, \end{aligned} \quad (91)$$

where (75a) is used. If δq and $\delta \mathbf{x}$ are symmetries of the lagrangian, i.e., $\delta L = 0$, we should consider $\delta t^2 = \theta$ a time-independent constant. In any case, δL is independent of $\delta \omega$, which means that any transformation in q is a lagrangian symmetry. The generator takes the form

$$G_g = -\pi \delta t^1 + \left(\frac{B}{2} \mathbf{x}^2 - v \right) \theta, \quad (92)$$

and the gauge transformations are finally given by

$$\delta q = \{q, G\} = \delta t^1, \quad (93a)$$

$$\delta \mathbf{x} = \{\mathbf{x}, G\} = \varepsilon \cdot \mathbf{x} \theta. \quad (93b)$$

C. The free Yang-Mills theory

Now let us turn to an example of field theory, the Yang-Mills (YM) theory without sources, described by the fundamental integral

$$I \equiv -\frac{1}{4} \int_{\Omega} d\omega F_{\mu\nu}^a F^{a\mu\nu}. \quad (94)$$

In our notation, Ω is 4-volume in a Minkowski space-time with metric $\eta = \text{diag}(+ - - -)$, and $d\omega$ is its volume element. The field strength is defined by

$$F_{\mu\nu}^a \equiv \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - g f^{abc} A_\mu^b A_\nu^c, \quad (95)$$

where g is a coupling constant and f^{abc} are the structure coefficients of an $\mathfrak{su}(n)$ algebra.

Let us define the covariant derivative

$$D_\mu^{ab} \equiv \delta^{ab} \partial_\mu - g f^{acb} A_\mu^c. \quad (96)$$

The field equations are

$$D_\mu^{ab} F^{b\mu\nu} = 0, \quad (97)$$

and the conjugated covariant momenta are given by

$$\pi^{a\mu\nu} \equiv F^{a\mu\nu}. \quad (98)$$

We remark that the gauge transformations related to the fundamental integral I are

$$\delta_g A_\mu^a = D_\mu^{ab} \epsilon^b, \quad (99)$$

where ϵ^a are the gauge parameters. These transformations leave I invariant up to a boundary term.

To get to the HJ formalism for singular field theories, it is necessary to choose a particular parametrization for the fields. Here, we will work with the ‘‘Galilean time’’ $t \equiv x^0$, choice that is known as the instant-form dynamics.²² In this case, the conjugated momenta are given by the projection of (98) in a unit 4-vector $u \equiv (1, 0, 0, 0)$:

$$\pi^{a\mu} \equiv \pi^{a\mu\nu} u_\nu = \pi^{a\mu 0} \equiv -F^{a0\mu}. \quad (100)$$

For $\mu = 0$ it gives

$$\pi^{a0} = 0, \quad (101)$$

while for $\mu = i$, where $\{i\} = \{1, 2, 3\}$, (100) gives an equation for the velocities \dot{A}_i^a ,

$$\dot{A}_i^a = D_i^{ab} A_0^b - \pi_i^a, \quad (102)$$

where we use the notation $\dot{A}_v^a \equiv \partial_0 A_v^a$.

We also have the (not symmetric) energy-momentum density:

$$H_{\mu\nu} \equiv -F_{\mu\gamma}^a \partial_\nu A_\alpha^\gamma + \frac{1}{4} \delta_{\mu\nu} F_{\alpha\beta}^a F_a^{\alpha\beta}. \quad (103)$$

With (100), (101), and (102), we have the canonical hamiltonian density $\mathcal{H}_c \equiv H_{\mu\nu} u^\mu u^\nu = H_{00}$, which up to a total divergence can be written by

$$\mathcal{H}_c = -\frac{1}{2} \pi^{ai} \pi_i^a - A_0^a D_i^{ab} \pi^{bi} + \frac{1}{4} F_{ij}^a F^{aij}. \quad (104)$$

Therefore, the system obeys two HJ equations,

$$H'_0 \equiv \pi_0 + \mathcal{H}_c = 0, \quad (105a)$$

$$\Phi^a \equiv \pi^{a0} = 0. \quad (105b)$$

For the theory to be integrable we calculate the PB between (105), using the fundamental relations $\{A_\mu^a(x), \pi^{bv}(y)\} = \delta^{ab} \delta_\mu^v \delta^3(x - y)$. The only non-zero bracket is given by

$$\{\Phi^a, H'_0\} = D_i^{ab} \pi^{bi}, \quad (106)$$

which implies non-integrability. This can be solved by imposing a new set of constraints

$$\Gamma^a \equiv D_i^{ab} \pi^{bi} = 0. \quad (107)$$

Now it is necessary the set (105), (107) to be in involution. The global sub-algebra of $\Gamma^a(x)$, calculated with the PB of the variables

$$\Gamma^a [h] \equiv \int_{\Sigma} d\sigma h(x) \Gamma^a(x) \quad (108)$$

is given by

$$\{\Gamma^a [h_1], \Gamma^b [h_2]\} = -gf^{abc} \Gamma^c [h_3], \quad (109)$$

where $h_3(x) = h_1(x)h_2(x)$, up to a boundary term in $\partial\Sigma$. In (108) the integration is performed in a 3-surface section Σ of Ω at constant t , whose volume element is $d\sigma$. The remaining PB is

$$\{\Gamma^a, H'_0\} = gf^{abc} A_0^b \Gamma^c. \quad (110)$$

The algebras (106), (109), and (110) indicate that the set (H'_0, Φ^a, Γ^a) is involutive with the PBs, therefore integrability is achieved. With the definition $T^b \equiv (i/g)\Gamma^a$, (109) becomes precisely the $\mathfrak{su}(n)$ algebra.

Now that we have the complete involutive system of HJ equations for the YM theory, the characteristic equations can be calculated by the fundamental differential

$$dF = \int_{\Sigma} d\sigma_y [\{F, H'_0(y)\} dt + \{F, \Phi^a(y)\} \omega^a(y) + \{F, \Gamma^a(y)\} d\lambda^a(y)]$$

where (t, ω^a, λ^a) is the set of independent variables, each one related to its respective generator. Again, because Γ^a come from integrability, the new set of independent variables λ^a is introduced.

For the variables A_μ^a we have

$$dA_\mu^a(x) = \int_{\Sigma} d\sigma_y \{A_\mu^a(x), H'_0(y)\} dt + \delta_\mu^0 d\omega^a(x) - \delta_\mu^i D_i^{ab}(x) d\lambda^b(x). \quad (111)$$

Since (t, ω^a, λ^a) are independent among themselves, we may write

$$\dot{A}_\mu^a(x) = \delta_\mu^i [D_i^{ab}(x) A_0^b(x) - \pi_i^a(x)], \quad (112a)$$

$$\frac{\delta A_\mu^a(x)}{\delta \omega^b(y)} = \delta^{ab} \delta_\mu^0 \delta^3(x-y), \quad (112b)$$

$$\frac{\delta A_\mu^a(x)}{\delta \lambda^b(y)} = -\delta_\mu^i D_i^{ab}(x) \delta^3(x-y). \quad (112c)$$

Equation (112b) indicates that $A_0^a = \omega^a$ plus an arbitrary function independent of the fields, which is a very property of a degenerate variable in the action. Equation (112c) indicates that the dynamics involves the variables λ^a in the form of the CT $\delta A_i^a = -D_i^{ab} \delta \lambda^b$, which depends on the variables A_i^a . Time evolution is given by (112a) that yields

$$\dot{A}_0^a = 0, \quad \dot{A}_i^a = D_i^{ab} A_0^b - \pi_i^a. \quad (113)$$

The equation for A_i^a reproduces (102), as expected.

For the variables $\pi^{a\mu}$ we have

$$d\pi^{a\mu}(x) = \int_{\Sigma} d\sigma_y \{\pi^{a\mu}(x), H'_0(y)\} dt + gf^{abc} \delta_i^\mu \pi^{bi}(x) d\lambda^c(x). \quad (114)$$

Again, independence of the parameters yields

$$\dot{\pi}^{a\mu}(x) = \delta_0^\mu \Gamma^a(x) - \delta_i^\mu [gf^{abc} A_0^c(x) \pi^{bi}(x) + D_j^{ab}(x) F^{bij}(x)], \quad (115a)$$

$$\frac{\delta \pi^{a\mu}(x)}{\delta \lambda^b(y)} = -gf^{abc} \delta_i^\mu \pi^{ci}(x) \delta^3(x-y). \quad (115b)$$

Equation (115a) represents time evolution:

$$\dot{\pi}^{a0} = \Gamma^a, \quad D_0^{ab} \pi^{bi} = -D_j^{ab} F^{bij}. \quad (116)$$

From the first equation, we notice that $\Gamma^a = D_i^{ab} \pi^{bi} = D_i^{ab} F^{bi0}$, that gives (97) for $\nu = 0$. The second equation is the field equation (97) for $\nu = i$, if they use the definition $\pi^{ai} = F^{ai0}$. Therefore, time evolution is equivalent to the field equations (97) for the YM field.

We now turn to the problem of finding the generator of the canonical transformations defined by Eqs. (112b), (112c), and (115b). They can be written by

$$\delta A_\mu^a = \delta_\mu^0 \delta \omega^a - \delta_\mu^i D_i^{ab} \delta \lambda^b, \quad (117a)$$

$$\delta \pi^{a\mu} = g f^{abc} \delta_\mu^i \pi^{bi} \delta \lambda^c. \quad (117b)$$

It is straightforward to show that

$$G_{CT} = \int_\Sigma d\sigma [\Phi^a \delta \omega^a + \Gamma^a \delta \lambda^a] \quad (118)$$

is the generator of the transformations (117).

The goal, now, is to find the generator of the gauge transformations. The first-order variation of the action (94) under a set of infinitesimal transformations $\delta x^\mu = \bar{x}^\mu - x^\mu$ and $\delta A_\mu^a = \bar{A}_\mu^a - A_\mu^a$ is given by

$$\delta I = \int_\Omega d\omega [\partial_\mu (F^{a\mu\nu} \delta A_\nu^a - H_\nu^\mu \delta x^\nu) + (\delta A_\nu^a - \delta x^\nu \partial_\nu A_\nu^a) D_\mu^{ab} F^{b\mu\nu}]. \quad (119)$$

As in the previous examples, the generator can be achieved with (119), under the transformations characterized by $\delta x^\mu = 0$ and $\delta A_\mu^a = \delta_\mu^0 \delta \omega^a - \delta_\mu^i D_i^{ab} \delta \lambda^b$. After some algebra, and the use of the identities $D_\mu^{ab} D_\nu^{bc} F^{c\mu\nu} = 0$, the following expression arises:

$$\delta L = \partial_\mu [(\delta \omega^a + D_0^{ac} \delta \lambda^c) F^{a0\mu}] + (\delta \omega^a + D_0^{ac} \delta \lambda^c) D_\mu^{ab} F^{b\mu 0}. \quad (120)$$

If the action is invariant under the transformations (117a), $\delta L = 0$ implies that the correct relationship between the independent variables of the theory is given by $\delta \omega^a = -D_0^{ab} \delta \lambda^b$. If we define $\Lambda^a \equiv -\delta \lambda^a$ as the gauge parameters, then $\delta \omega^a = D_0^{ab} \Lambda^b$. Of course, the transformation in $\pi^{a\mu}$ has no analogous in the lagrangian picture. In this case, the generator (118) becomes

$$G_g = \int_\Sigma d\sigma [\Phi^a D_0^{ab} \Lambda^b - \Gamma^a \Lambda^a] = \int_\Sigma d\sigma \pi^{a\mu} D_\mu^{ab} \Lambda^b, \quad (121)$$

up to a boundary term. To check this generator, we calculate

$$\delta_g A_\mu^a(x) = \{A_\mu^a(x), G_g\} = D_\mu^{ab}(x) \Lambda^b(x), \quad (122)$$

which are in fact the gauge transformations (99) of the theory. We may perform a further calculation,

$$\delta_g \pi_\mu^a(x) = \{\pi_\mu^a(x), G_g\} = -g f^{abc} \pi_\mu^b(x) \Lambda^c(x), \quad (123)$$

which agrees with (117b), considering the HJ equation $\pi_0^a = 0$.

VI. FINAL REMARKS

As a continuation of the work,¹⁵ we analyzed the integrability of constrained systems within the Hamilton-Jacobi formalism, and studied how this approach links complete sets of involutive HJ equations with canonical and lagrangian point (gauge) symmetries, as named by Dirac,⁵ of a fundamental integral. Now let us highlight the main script of this study.

According to Carathéodory's "complete figure" applied to singular systems, the necessary and sufficient condition for the existence of an extreme configuration of the action (1) is the existence of a function $S(t^\alpha, q^a)$ of the configuration space that obeys the conditions (11b), and is a complete solution of the set of HJ equations (11a). If the independent variables t^α are linearly independent among themselves, the set of HJ equations are related to the characteristic equations (12), whose solutions are trajectories $\xi^A = \xi^A(t^\alpha)$ in a reduced phase space $\mathbf{T}^*\mathbb{Q}_P$, parametrized by t^α .

The necessary and sufficient conditions for the linear independence of the parameters t^α , and therefore for the existence of the characteristic equations themselves, happen to be the same conditions for the existence of a complete solution of the HJ equations. In Sec. III, we introduced the Frobenius' theorem: a system of first-order PDEs $H'_\alpha(t^\beta, \pi_\beta, q^a, p_a) = 0$ is completely integrable if, and only if, the functions H'_α obey the Frobenius' integrability conditions (23). Sufficient conditions for integrability can, on the other hand, be generalized to a Poisson algebra (25). Since $H'_\alpha = 0$ are also canonical constraints, integrability demands these constraints to be in involution with the PBs. Therefore, a complete set of integrable HJ equations are also a set of involutive constraints.

If $H'_\alpha = 0$ is a complete set of involutive constraints, Sec. IV shows that they are generators of active canonical transformations in the complete phase space $\mathbf{T}^*\mathbb{Q}_{N+1}$. This is done by building the symplectic structure: first, we introduced the vector fields X_α , related to the functions H'_α . With integrability, the symplectic 2-form ω splits in two 2-forms ω_p and a , Eq. (31), such that a is identically a null form. In this case, we have shown a known result that singular systems have a degenerate symplectic structure, whose vector fields X_α form a basis of the null space of ω , as can be seen by the relations (36) and (37). Infinitesimal transformations (38), generated by the involutive constraints, preserve the symplectic structure (42), and therefore Liouville's theorem is implied. Moreover, a canonical action defined in (43) is invariant under the so called characteristic flows (46). For constant structure coefficients, the Poisson algebra (25) implies the Lie algebra (29), and a Lie group of transformations can be built from the Lie algebra of the characteristic vector fields.

The connection between the characteristic flows and lagrangian point (gauge) transformations is discussed in Sec. IV D. Point transformations (47) are CFs in which $\delta t = 0$, i.e., time-independent canonical transformations. However, to be canonical, the transformations themselves must be restricted to $\mathbf{T}^*\mathbb{Q}_p$, since their generators H'_z must obey their own Poisson algebra (48). Because point transformations are Noether symmetries, characterized by a set of gauge parameters ϵ , the Lie equation (51) can be used to relate δt^α to this set. In general, it results in linear dependency between the independent variables t^α , as shown directly in the examples of Sec. V. We notice that no analogous of Dirac's conjecture is needed in these results, but the generator of the characteristic flows depends on a complete set of involutive constraints H'_z , as a direct result of Frobenius' theorem. The dependency between the parameters t^α , on the other hand, is used to build the generator of gauge transformations directly from the generator of the characteristic flows.

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