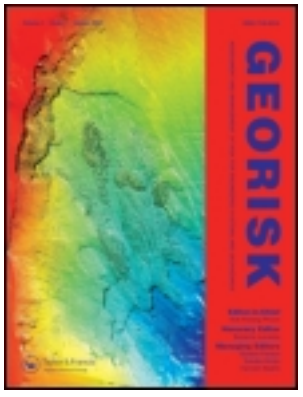

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Approximate up-scaling of geo-spatial variables applied to deep foundation design

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Approximate up-scaling of geo-spatial variables applied to deep foundation design

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We present a series of simple approximate methods for up-scaling the cumulative distribution function of spatially correlated variables by using an effective number n_e of independent variables. Methods are based on the property of distribution permanence of the gamma and inverse Gaussian distributions under averaging, bootstrap sampling and expansions about the normal and gamma distributions. A stochastic simulation study is used to validate each method, and simple parameters are defined to identify respective ranges of applicability. A practical example is presented where core sample rock strength data are up-scaled to shaft size for probabilistic (risk-based) deep foundation design. Supplemental material is available online.

Keywords: probability of failure; reliability; change of support; geostatistics; Edgeworth; gamma expansion; bootstrap; inverse Gaussian

1. Introduction

Spatial scale is one of the most fundamental parameters in science and engineering. However, data are typically not defined on the same scale of support as required for subsequent processing and decision-making, which raises the question about the effect of changing scales. This is known as the 'change of support' problem, and for the particular case of changing support from small to large, as the 'up-scaling' problem. The topic has received considerable attention in the (geo-) statistical literature (Chilés and Delfiner 1999, Gotway and Young 2002) relating to many fields of application including mining/petroleum engineering, hydrology, agriculture, etc. A classic example is the use of core sample data (1 cm scale) in the determination of total recoverable ore tonnage in a reserve by estimating the cumulative ore content of mining blocks (10 m scale) above a certain cut-off level. Many times core sample data also need to be used as input to numerical models of a desired domain size and limited number of discrete cells (possibly at a 1–100 m scale).

The present work is motivated by a practical problem in geotechnical engineering – in particular, reliability- (or risk-) based deep foundation design (e.g. for bridges; Phoon *et al*. 2003, AASHTO 2004), where it is the goal to assure compliance with a target (maximum permissible) probability of failure p_f of a foundation in order to limit the level of risk of potential damages (e.g. collapse or excessive settlement). For this purpose, both design load Q and foundation resistance R are generally regarded as random variables such that the design goal may be expressed mathematically as

$$P[R < Q] \leq p_f, \tag{1}$$

where P[ ] denotes the probability (risk) of the event in brackets (load exceeding resistance) to occur. Evaluation of Equation (1) requires knowledge of the exact probability density functions (pdfs) and/or cumulative distribution functions (cdfs) of R and Q. Limiting attention to the resistance side, the sources of uncertainty affecting cdf(R) may be categorised into three principal classes: (1) spatial variability of ground properties, (2) measurement errors and (3) uncertainty in data transformation (Phoon and Kulhawy 1999a, 1999b). A method to estimate a lump value of all uncertainty types is based on past experience and the compilation and analysis of comprehensive load test databases, which allow for assessment of prediction error distributions (and possibly model calibration) for different combinations of site conditions, prediction and construction methods (Zhang *et al*. 2001, 2008, Haldar and Babu 2008). However, inherent shortcomings with this method are that it does not offer an explicit possibility to account for site-specific data and that the characteristics of a site/job have to be matched with a sufficient number of corresponding observations from the past. As an alternative, general approaches have been proposed that evaluate the contributing

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sources of uncertainty separately and then combine them according to appropriate physical and statistical laws (Phoon and Kulhawy 1999a, 1999b, Foye *et al.* 2006). Approaches that explicitly account for spatial variability are not very abundant and consider shallow foundations in two (Paice *et al.* 1996, Fenton and Griffiths 2002, 2003, Popescu *et al.* 2005, Babu *et al.* 2006) and three (Fenton and Griffiths 2005) dimensions. Fenton *et al.* (2005, 2008) relate their previous results to Load and Resistance Factor Design (LRFD) and investigate effects of data in the vicinity of a shallow foundation to reduce resistance uncertainty. Furthermore, Fenton and Griffiths (2007) present a preliminary finite element study for a single-object deep foundation subject to vertically variable ground properties.

The present work develops methods for investigating the effects of spatial variability on cdf(R), where R is considered as the total ultimate (i.e. deformation independent) axial resistance of a drilled shaft due to side friction and neglecting end bearing (which is common practice with design in Florida limestone, for example; FDOT 2006). Thus, as illustrated in Figure 1, R is known to be equal to the integral of local ground (i.e. rock or soil) strength (or unit side friction) q over the lateral shaft surface area A_s (typically a cylinder; Klammler *et al.* 2010):

$$R = \int_{A_s} q dA. \tag{2}$$

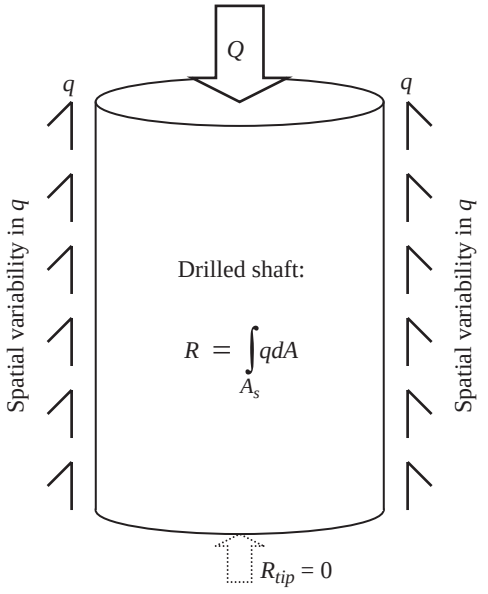

Figure 1. Schematic of ultimate (and, hence, deformation independent) axial shaft resistance R due to side friction only. Tip resistance R_{tip} is neglected according to design practice in Florida limestone (FDOT 2006).

However, in early design stages, exact shaft locations are unknown (i.e. in some sense random) and local ground strength data are available, for example, based on core sample analysis from a number of (somehow also randomly located) borings on a site. The resulting problem of Equation (2) is to up-scale the observed cdf(q) at (quasi-) point support to cdf(R) at a support equal to the lateral shaft surface for subsequent use in Equation (1). Note that the problem as stated does focus on transitions in spatial scale; however, support sizes of both q and, consequently, R are considered larger than the representative elementary volume (REV) of what may be considered local strength or unit side friction in rock/soil (e.g. measured with samples obtained from drilling cores). The term 'quasi-point support' for the scale of q will be merely used to designate a scale much smaller than the scale of spatial variability of q (i.e. the separation distance beyond which local values of q become uncorrelated).

Klammler *et al.* (2010) studied the problem of Equation (2) with log-normal up-scaling (Isaaks and Srivastava 1989) using the well-known approximation that the sums (or averages) of log-normal variables are again log-normal. However, this approach is limited to log-normal distributions of q, which may be prohibitive in practice. Approximate analytical models for up-scaling of spatially correlated variables of arbitrary distributions include affine and indirect log-normal corrections (Isaaks and Srivastava 1989) and are recommended for use only when reduction in variance is below 30%, since they do not honor the Central Limit Theorem (Deutsch 2002). More sophisticated approaches, including disjunctive kriging, the discrete Gaussian and mosaic methods, are described in Cressie (1993) and Chilés and Delfiner (1999), while stochastic simulation (Deutsch 2002) is a numerical alternative based on random field generation. However, application of the latter methods is limited in geotechnical design practice due to their elevated complexity, and critical underlying assumptions may not always be based on sufficient data. For a more detailed overview of up-scaling techniques see Gotway and Young (2002).

From this, we detect a lack of spatial up-scaling methods which are both appropriate for any degree of variance reduction (i.e. spatial averaging) and which are simple enough for practical implementation by engineers with limited (geo-) statistical background. By using the concept of effective numbers of independent variables the present paper develops a series of simple approximate methods, which combine the variance reduction principle due to spatial averaging from geostatistics with distributional properties from classical statistics. Averaging (or

summing) of independent variables is a thoroughly studied field (Petrov 1975), and the concept of effective numbers of independent data has been explored previously in many different ways (Bayley and Hammersley 1946, Kitanidis 1997, Deutsch 2004, Pardo-Iguzquiza and Dowd 2004). Two parametric (gamma and inverse Gaussian) and three non-parametric (bootstrap-based and expansions about the normal and gamma distributions) methods are developed in the present work and validated against results from stochastic simulations. The existing method of parametric log-normal up-scaling is also included in the validation study to obtain a comparative evaluation of its performance under different conditions and to study its range of applicability. Limiting assumptions in the present and previous approaches are stationarity of the underlying random process and that the variogram is sufficient to describe spatial variability (Deutsch 2002).

2. Variance reduction and effective number of independent variables

In concordance with the problem of shaft resistance posed in Equation (2), we focus on arithmetic averaging of spatially correlated random functions (regionalised variables; Journel and Huijbregts 1978), which can be expressed in a general form by

$$Z_v(u) = \frac{1}{v}\int_v Z_0(x)dx, \tag{3}$$

where x is a coordinate vector and $Z_0(x)$ a regionalised variable defined on point support and described by a cumulative distribution function (cdf) with expectation μ_0 and variance σ_0^2 as well as a variogram or, equivalently, a spatial covariance function $C(h)$, with h being a spatial separation vector between two locations x and x'. $Z_v(u)$ is the corresponding effective (i.e. arithmetically averaged) parameter for support size $v = \int_v dx$ centred on coordinate vector u.

Note that v can be a joint or disjoint domain in one, two, or three (or more) dimensions. Equation (2) is a particular case of Equation (3) for $Z_0 = q$, $v = A_s$ and $Z_v = R/A_s$, the latter being equivalent to mean unit side friction over the lateral shaft surface. As such, the present work considers cdf and $C(h)$ of $Z_0(x)$ as reliably known, while the location u of v is random (i.e. unknown or distant from available data). Hence, there is no conditioning to data (unconditional up-scaling) and $Z_0(x)$ can be abbreviated to Z_0 as well as $Z_v(u)$ to Z_v, whose cdf of expectation μ_v and variance σ_v^2 is sought.

It is well known (Deutsch 2002) from the Central Limit Theorem that up-scaling of linearly averaging variables entails constant expectation ($\mu_v = \mu_0$), while variance, skewness and higher-order cumulants decrease. The simplest case is that of Gaussian random fields where up-scaled distributions are again Gaussian; however, distributions are not generally preserved between scales. The reduced variance σ_v^2 is found from (Deutsch 2002)

$$\sigma_v^2 = \frac{1}{v^2}\int_v\int_v C(x-x')dxdx'. \tag{4}$$

Equation (4) only depends on $C(h)$ and v and is valid independent of the underlying distribution type. If Equation (4) is applied to a finite number n of uncorrelated random variables it reduces to $\sigma_v^2 = \sigma_0^2/n$, which is the relationship for the variance of a mean estimate in classical statistics requiring the assumption of 'i.i.d.' (independent and identically distributed) variables. By following a concept briefly introduced in the opposite context of estimating a process mean from limited data by Kitanidis (1997) or Deutsch (2004), an effective number of independent variables n_e can be defined as

$$n_e = \frac{\sigma_0^2}{\sigma_v^2}, \tag{5}$$

which expresses the number of independent random variables that are subject to the same amount of variance reduction when averaged as spatial averaging of a regionalised variable Z_0 with covariance structure $C(h)$ over a domain v.

3. Parametric methods

Data distributions encountered in science and engineering are mostly non-negative and positively skewed with a rather long tail to the right. The log-normal is a two-parameter distribution that has been widely applied to fit these observations before and after up-scaling (direct log-normal up-scaling; Isaaks and Srivastava 1989, Vargaz-Guzmán 2005). However, it is well known that averages of even independent log-normal variables are not exactly log-normally distributed (Santos Filho *et al.* 2005). In contrast, other non-negative and positively skewed two-parameter distributions like the gamma and inverse Gaussian appear to be more appropriate for this type of 'direct' up-scaling than the log-normal distribution, as they possess the property of strict distributional permanence under arithmetic averaging of independent variables (Krishnamoorthy 2006). For example, the average of two or more independent and identically distributed gamma variables is again gamma distributed, however with different parameters. This preservation of distribution type with

up-scaling does not strictly hold for spatially correlated variables, but, motivated by the widely used approximate direct log-normal up-scaling, the present work proposes approximate direct gamma and inverse Gaussian up-scaling. With this, in analogy to the classic direct log-normal (*LN*) up-scaling, given that Z_0 may be assumed to be either gamma (*GA*) or inverse Gaussian (*IG*), the up-scaled cdf is approximated by the same distribution type. This significantly widens the range of applicability of the simplest 'direct' (i.e. using approximate distribution permanence) up-scaling techniques, which honor the Central Limit Theorem for large degrees of variance reduction (such as *LN*, *GA* and *IG* also approach normality as the variance decreases).

For convenience, expressions for pdfs and cdfs of these distributions are given in the online supplemental material. Table 1 summarises the meanings of the two parameters (p_1, p_2) for each distribution (including *LN*) and the relationships to distribution expectation μ and coefficient of variation $CV = \sigma/\mu$ (in order to remain close to common geotechnical practice we prefer the use of CV to σ or σ^2). Direct up-scaling is simply performed by applying $\mu = \mu_v = \mu_0$ and $CV = \sigma_v/\mu_v$ [using Equation (4)] in columns 2 and 3 to estimate distributional parameters p_1 and p_2 such that the cdf and pdf of Z_v are fully defined. Distributional fits/tests can be performed to evaluate which (if any) distribution type best fits Z_0. However, the last column of Table 1 provides simple expressions of the coefficient of skewness *sk* of each distribution in terms of CV. The three distributions may be ordered according to increasing *sk* for a given CV, thus suggesting the ratio sk/CV as an efficient field parameter to discern (up to third order) which distribution may be most adequate for given data. $sk/CV \approx 2$ suggests a gamma fit, $sk/CV \approx 3$ an inverse Gaussian and $sk/CV \approx 3 + CV^2 > 3$ a log-normal fit.

Table 1. Parameters of *GA*, *IG* and *LN* distributions (Krishnamoorthy 2006).

	p_1	p_2	*sk*
Gamma	μCV^2 (scale)	$\frac{1}{CV^2}$ (shape)	$2CV$
Inverse Gaussian	μ (mean)	$\frac{\mu}{CV^2}$ (shape)	$3CV$
Log-normal	$\ln\frac{\mu}{\sqrt{1+CV^2}}$ (log-mean)	$\sqrt{\ln(1+CV^2)}$ (log-std. dev.)	$3CV+CV^3$

4. Non-parametric methods

4.1. Bootstrap

Stochastic simulation is a non-parametric method using the cdf and the variogram of Z_0 to generate realisations of random fields, which are averaged over v to obtain a discrete cdf of Z_v in a Monte Carlo sense. However, as initially mentioned, the fact that random field realisations have to be drawn from a population of given cdf and variogram involves a significant degree of computational complexity and software operational skills for (geo-) statistically untrained engineers (the risk of blindly trusting computer outputs is not acceptable). The first non-parametric method proposed here avoids the generation of random field realisations by making use of the effective number of independent variables n_e defined in Equation (5). That is, $C(h)$ is known such that σ_v can be determined from Equation (4) followed by n_e from Equation (5). Instead of arithmetic averaging over v in each of N random field realisations, arithmetic averaging over n_e independent samples randomly drawn from the cdf of Z_0 is performed N times to arrive at a discrete approximation for the cdf of Z_v. If N is large enough both the permanence of mean $\mu_v = \mu_0$, the variance reduction criterion of Equation (4) as well as the normalisation of the cdf of Z_v due to the Central Limit Theorem are satisfied by this method. From Equations (4) and (5) it is evident that n_e is generally not an integer, and a solution for how to draw a non-integer number of samples from a population is given in the appendix. Due to the similarities of this approach with the bootstrap method (Efron and Tibshirani 1998) it is hereafter referred to as 'bootstrap'.

4.2. Edgeworth expansion

The second and third non-parametric approaches developed here are based on series expansions of cdfs about normal and gamma distributions respectively, where only a certain number of lower-order moments or cumulants of a cdf are retained. This will result in relatively simple results and is appropriate for most practical situations where limited data do not allow for reliable inference of higher-order moments. The term 'non-parametric' is justified by the fact that no distribution type is assumed for Z_0. If we denote the ith centralised moment and the ith cumulant of a random variable Z of mean μ and variance σ^2 by μ_i and κ_i, respectively, the following relationships apply (Hall 1992): $\kappa_1 = \mu_1 + \mu = \mu$, $\kappa_2 = \mu_2 = \sigma^2$, $\kappa_3 = \mu_3$ and $\kappa_4 = \mu_4 - 3\mu_2^2$, where $\mu_i = \mathrm{E}[(Z-\mu)^i]$ with E[ ] being the expectation operator. Standardised cumulants are

frequently used as κ_i/σ^i and are known as coefficient of skewness sk if $i=3$ and as excess kurtosis ek if $i=4$. Important properties of cumulants for the present work are that (1) if Z is scaled by a factor c, then respective cumulants are scaled by factors c^i, i.e. $\kappa_i(cZ)=c^i\kappa_i(Z)$ and (2) cumulants of sums of independent random variables are equal to the respective sums of cumulants, i.e. $\kappa_i(Z_1+Z_2)=\kappa_i(Z_1)+\kappa_i(Z_2)$. From this, the cumulants of a mean

$$Z_v=\frac{1}{n}\sum_{j=1}^{n}Z_j$$

with Z_j independent and identically distributed are obtained as $\kappa_i(Z_v)=\kappa_i(Z_j)/n^{i-1}$ giving further $sk(Z_v)=sk(Z_j)/n^{1/2}$ and $ek(Z_v)=ek(Z_j)/n$.

For an expansion about the normal distribution, Edgeworth series are known to provide asymptotic approximations to a general class of cdfs and, in the case of averaging n independent variables Z, a general form limited to effects of skewness and excess kurtosis is (Hall 1992)

$$F(t)=F_N(t)-\left\{\frac{sk}{6\sqrt{n}}(t^2-1)+\frac{1}{24n}\left[ek(t^3-3t)+\frac{sk^2}{3}(t^5-10t^3+15t)\right]\right\}f_N(t). \quad (6)$$

Here $t=(Z_v-\mu)\sqrt{n}/\sigma$ is the standardised mean Z_v with $F(t)$ being the approximate cdf of t and $F_N()$ and $f_N()$ being the standard normal cdf and pdf, respectively, as defined in the online supplemental material. sk and ek are used for $sk(Z)$ and $ek(Z)$, respectively. While for $n=1$ no averaging takes place and Equation (6) reduces to approximating the cdf of Z, for large n the corrective terms in the curly brackets vanish and $F(t)$ approaches $F_N(t)$ as required by the Central Limit Theorem. Similarly, for $sk=ek=0$ skewness and excess kurtosis of Z agree with those of the normal distribution and the second-order Edgeworth expansion reduces to $F(t)=F_N(t)$. In analogy to the bootstrap approach, Equation (6) is applied to the up-scaling problem of the spatially correlated variable Z_0 in an approximate way by using $Z=Z_0$ and $n=n_e$ from Equation (5).

4.3. Gamma expansion

Since low-order expansions about the symmetric normal distribution can be expected to become inaccurate for averages of strongly non-normal variables and insufficient normalisation due to averaging, expansion about the gamma distribution is proposed for positively skewed non-negative variables. Bowers (1966) gives a solution for the approximation $F(w)$ of a general cdf (without averaging) by a non-asymptotic series of gamma functions in the form of

$$\begin{aligned}F(w)=F_{GA}(w,p_2)&-A[f_{GA}(w,p_2+1)\\&-2f_{GA}(w,p_2+2)+f_{GA}(w,p_2+3)]\\&+B[f_{GA}(w,p_2+1)-3f_{GA}(w,p_2+2)\\&+3f_{GA}(w,p_2+3)-f_{GA}(w,p_2+4)]\end{aligned} \quad (7)$$

where terms containing higher-order moments than kurtosis were truncated. The expressions for $F_{GA}(w,p)$ and $f_{GA}(w,p)$ are given in the online supplemental material and are the gamma cdf and pdf, respectively, of shape parameter p for the standardised variable $w=Z\mu/\sigma^2$. This transformation assures that the mean and variance of w are both equal to $\mu^2/\sigma^2=1/CV^2$, which are then matched by $F_{GA}(w,p_2)$ with $p_2=1/CV^2$ (Table 1). The coefficients A and B are functions of the third and fourth centralised moments of w as well as of p_2. From the relationships between centralised moments and cumulants given above and knowing that $\kappa_3(GA)=2p_2$ and $\kappa_4(GA)=6p_2$ [with $\kappa_3(GA)$ and $\kappa_4(GA)$ being the third and fourth cumulants of $F_{GA}(w,p_2)$, respectively] one can rewrite the expressions of Bowers (1966) for A and B as

$$A=\frac{1}{6}(\kappa_3(w)-\kappa_3(GA)) \quad (8)$$

and

$$B=\frac{1}{24}[(\kappa_4(w)-\kappa_4(GA))-12(\kappa_3(w)-\kappa_3(GA))]. \quad (9)$$

This shows that corrective terms after $F_{GA}(w,p_2)$ in Equation (7) are weighted by simple functions of the differences between the third and fourth cumulants of the standardised variable w and $F_{GA}(w,p_2)$, the gamma distribution about which the expansion is built. From the relationship between Z and w, one obtains $\kappa_3(w)=sk(Z)/CV^3$ and $\kappa_4(w)=ek(Z)/CV^4$. In addition, with $\kappa_3(GA)=2/CV^2$ and $\kappa_4(GA)=6/CV^2$ A and B can be expressed in terms of parameters of Z only.

For the present purpose, Equations (7–9) are generalised to allow for approximating the cdf of some mean Z_v over n independent and identically distributed variables Z by using $w=Z_v\mu/\sigma_v^2=Z_v\mu n/\sigma^2$, such that $p_2=\mu^2 n/\sigma^2$ is equal to the common value of mean and variance of w. Taking advantage of the properties of cumulants under scaling and averaging as already discussed, $\kappa_3(w)=sk(Z)n/CV^3$ and $\kappa_4(w)=ek(Z)n/CV^4$ are used in Equations (8) and (7), to account for the effects of skewness and kurtosis reduction due to averaging; thus,

$$A = \frac{n}{6CV^2}\left(\frac{sk}{CV} - 2\right) \tag{10}$$

$$B = \frac{n}{24CV^2}\left[\left(\frac{ek}{CV^2} - 6\right) - 12\left(\frac{sk}{CV} - 2\right)\right], \tag{11}$$

where sk and ek equate to $sk(Z)$ and $ek(Z)$, respectively. For $n = 1$ these expressions reduce to Equations (8) and (9) for no averaging and $F(w)$ in Equation (7) becomes an approximation of the cdf of Z. For large n the coefficients A and B increase; however, proportionally large values of p_2 lead to a cancelling out of terms within the brackets of Equation (7), and the basic expansion term $F_{GA}(w,p_2)$ approaches a normal distribution in agreement with the Central Limit Theorem. Independent of n, for $k_3(w) = k_3(GA)$ and $k_4(w) = k_4(GA)$, the first four cumulants of the cdf of w and $F_{GA}(w,p_2)$ are identical leading to $A = B = 0$ and $F(w) = F_{GA}(w,p_2)$. This reduces the approach to the preceding parametric one of directly fitting a gamma distribution to a distribution of Z_v with known mean and variance. In this sense, the non-parametric gamma expansion extends the parametric gamma method by accounting for effects of skewness and kurtosis in addition to simply mean and variance. The parametric (direct) gamma method implicitly assumes that skewness and kurtosis are defined by the gamma distribution and not the properties of Z. Equations (7), (10) and (11) are applied to the up-scaling problem of the spatially correlated variable Z_0 in an approximate way by using $Z = Z_0$ and $n = n_e$ from Equation (5). Note, finally, that in contrast to the bootstrap method both the Edgeworth and gamma expansion methods can directly handle non-integer values of n_e.

5. Simulation study and results

The principal approximation involved in the approaches presented is that averaging of correlated variables is substituted by averaging of an effective number of independent variables. Moreover, the classic direct log-normal method assumes approximate log-normality of means of independent log-normals; also, the non-parametric Edgeworth and gamma expansion methods are limited to a finite number of terms and convergence is not always guaranteed (Hall 1992). To validate these approximations a comprehensive simulation study is performed, in which outcomes of the present approaches are compared to up-scaling by stochastic simulation of 10,000 random field realisations. Gaussian random fields are generated by the method of LU-decomposition with subsequent inverse normal score transformation (Goovaerts 1997) to achieve a target distribution. A total of 16 test distributions including bimodal, truncated, negatively skewed and discrete distributions of $CV \leq 2$ and $-0.5 < sk \leq 14$ are used in the simulation study, which is described in detail in the online supplemental material.

To investigate the influence of different correlation patters inside the spatial averaging domains, the simulation study considers two-dimensional averaging in combination with an isotropic and an anisotropic spatial correlation structure. Hereby, averaging domain sizes are assumed to be squared and range from much smaller to much larger than the spatial correlation range. In addition, spatial averaging over a disjoint domain of two points separated by half the correlation range is considered as a type of worst-case scenario, where correlation is neither zero (i.i.d. case) nor perfect (no spatial averaging), but approximately 0.5.

It is uniformly observed that direct (parametric) *GA*, *IG* and *LN* approximations perform well if the underlying population distribution is of the same type or sufficiently similar (e.g. *IG* and *LN* for $CV \leq 0.5$). This confirms that, to the level of accuracy applied in this study (10,000 realisations), spatial averaging over a spatially correlated variable may be accurately approximated by simple averaging of n_e independent variables, for which an exact solution is known in the case of *GA* and *IG* distributions. For the *LN* the additional assumption of log-normality of means of independent variables is also seen to have a negligible effect on the results presented. For large numbers of n_e, results become acceptable independent of the combination of test/approximation distribution, thus confirming the expected convergence to normality from the Central Limit Theorem. From this and additional results using discrete test distributions (being bimodal, negatively skewed and/or truncated) it is further seen that the choice and appropriateness of direct *GA*, *IG* or *LN* up-scaling may be well evaluated by the proposed ratio sk/CV according to Table 1.

Among all methods studied, the bootstrap appears to be almost uniformly the best method. However, if the population distribution is given in a discrete form and n_e is close to 1, then artifacts of the discrete population cdf may remain after bootstrapping. This effect may be reduced by drawing bootstrap samples from a smoothed distribution. In the present study the discrete population cdfs of the non-parametric test distributions are linearly interpolated for both bootstrapping and random field simulation (without tail extrapolation). As to be expected, the performance of the Gaussian-based Edgeworth expansion method deteriorates rapidly as CV and sk increase and n_e decreases. Moreover, the improvement between first-order [Equation (6) neglecting the

term of order $1/n$; correcting for primary effects of skewness] and second-order [full Equation (6); correcting for primary effects of kurtosis and secondary effects of skewness; Hall 1992] expansions is quite limited or non-existent. However, simulation results and Equation (6) suggest the definition of a parameter $\Delta sk_N = |sk|/\sqrt{n_e} = |sk(Z_v)|$ as a measure of convergence to normality and applicability of the Edgeworth approximation. The expression (t^2-1) $f_N(t)/6$ in Equation (6) possesses a maximum absolute value of approximately 0.4/6, from which it is inferred that for $\Delta sk_N \le 0.15$ the first-order correction term is smaller than 1% for all t and the cdf of t can be assumed to be accurately approximated by the normal distribution $F_N(t)$. On the other hand, simulation results indicate that the Edgeworth method delivers acceptable results only for the approximate range of $\Delta sk_N \le 1$. Failures of the Edgeworth method for $\Delta sk_N \le 1$ are possible as demonstrated by test distributions, for example, whose bimodal nature is too complex to be captured by a single parameter based on skewness.

The gamma expansion method, as expected, improves on the Edgeworth method for positively skewed test distributions, with little or no improvement from a one-term [Equation (7) with $B=0$] to two-term [full Equation (7)] expansion. For absolute values of skewness close to 0, the gamma expansion performs worse than the Edgeworth expansion; however, it still outperforms the parametric approximations. In general, the gamma expansion method is seen to consistently improve upon the parametric gamma approximation, except for cases where the parametric gamma approximation itself is in large error and the respective expansion becomes unstable. In analogy to Δsk_N Equations (7) and (8) may be used to define a parameter $\Delta sk_{GA} = |sk(Z_v) - sk(GA)| = |sk - 2CV|/\sqrt{n_e}$ expressing the difference in skewness between Z_v (or equivalently w) and the gamma fit $F_{GA}(w,p_2)$. The simulation study indicates that the gamma expansion method delivers acceptable results in the approximate range of $\Delta sk_{GA} \le 0.5$ and that the improvement of the gamma expansion over the simple parametric gamma fit becomes negligible for $\Delta sk_{GA} \le 0.15$.

6. Practical example

Based on FDOT (2009), Klammler *et al.* (2010) present a data analysis and case study of a site in Florida, where drilled shafts are considered for bridge foundation and 136 core sample rock strength data are available from six borings (see online supplemental material). The summary statistics of the data are $\mu = 2.04$ MPa, $CV = 0.50$ and $sk = 0.50$ with a spatial covariance function consisting of two components: 80% of the total variance has vertical and horizontal correlation ranges of 1.5 and 4.5 m, respectively (geometric anisotropy), and the remaining 20% of the total variance is only contained in the horizontal direction with a range of 4.5 m (zonal anisotropy; vertical range very large). These are parameters obtained from variogram analysis (e.g. Isaaks and Srivastava 1989) of the core sample data as performed by Klammler *et al.* (2010). Assuming shaft diameter and length of 1.2 and 9 m, respectively, Klammler *et al.* (2010) apply their charts to graphically find a variance reduction factor $\alpha = 0.23$ (see their Table 1, row for γ_A), which is directly converted into an effective number of independent variables by $n_e = 1/\alpha = 4.33$. With this, the methods presented here are applied to estimate the distribution of a shaft's side friction resistance and compared to results of full stochastic simulations [10,000 realisations; shaft discretised into 20 (circumference) $\times$ 50 (length) elements].

Results in terms of maximum absolute differences in per cent cdf values between approximations and stochastic simulation are as follows: direct *GA*: 1.4; direct *IG*: 2.9; direct *LN*: 2.9; bootstrap: 0.9; Edgeworth first order: 0.9; gamma one term: 1.0. Edgeworth second order and gamma expansion with two terms performed similar to or slightly worse than their simpler counterparts. These numbers may be compared to a benchmark of 2.3, which is the maximum absolute difference in cdf values between two samples of 10,000 realisations that is not exceeded with a probability of 99% according to the Kolmogorov–Smirnov test. From this it is seen that direct *GA*, bootstrap, Edgeworth and gamma expansions produce results within the margins of error of the stochastic simulation. Using the proposed ratio $sk/CV = 1$ and the last column of Table 1 predicts the evidently better performance of the direct *GA* method over direct *IG* or *LN* (as used in Klammler *et al.* 2010). Moreover, the proposed parameters $\Delta sk_N = \Delta sk_{GA} = 0.24$ (coincidentally the same) are below their limits found of 1 and 0.5, respectively, which is reflected by the good performance of the expansion methods. Furthermore, Figure 2 shows the lower tails of cdf(R) from selected methods. For a given probability (risk) p_f of failure a maximum permissible load Q (for simplicity of illustration assumed to be deterministic, i.e. without uncertainty) is obtained as $Q = R$ when entering the chart with cdf(R) $= p_f$. For example, for $p_f = 0.01$, Q from direct *GA*, bootstrap and gamma one term are within approximately 1 MN (3%) of $Q \approx 36$ MN from the full simulation result; for $p_f = 0.001$

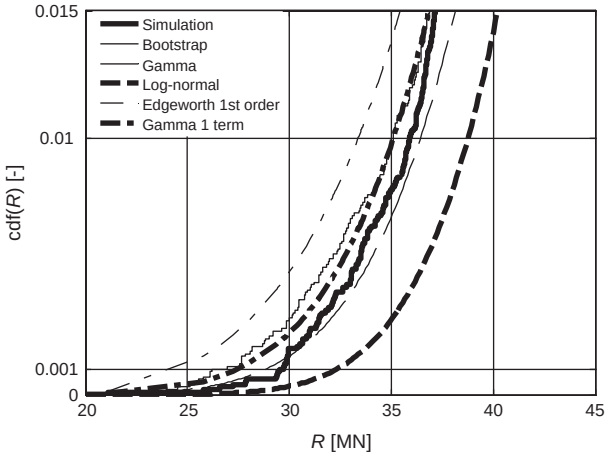

Figure 2. Lower tails of up-scaled resistance distributions for field example. For deterministic loads Q the probability of failure is equal to cdf(R) at $R = Q$.

deviations somewhat increase (illustrating the increasing difficulty when working in extreme portions of distribution tails); however, deviations are seen to occur consistently towards the conservative side in the present case. While Edgeworth first order results in more conservative Q, direct LN (as well as IG; not shown) up-scaling is consistently unconservative. However, as already stated, the ratio criterion sk/CV may be applied in practice to identify the latter two methods as potentially worse for the present data than the direct GA method (or its expansion).

7. Summary and conclusion

Risk- (or reliability-) based design of deep foundations aims at not exceeding a prescribed maximum probability of failure. For mathematical treatment of the problem according to Equation (1) and a given deterministic (zero uncertainty) or random load it is required to know the probability distribution of random resistance. The properties of the resistance distribution are affected by different sources of uncertainty. One of these sources is spatial variability, which is investigated in the present work in the context of ultimate (deformation independent) axial resistance of drilled shafts due to side friction, where local strength data is available at a quasi-point support (e.g. core samples). For this purpose, we present a series of simple approximate methods for up-scaling of linearly averaging variables based on a combination of classical and geostatistical principles that are linked by the use of an effective number n_e of independent variables. The number n_e is equivalent to a factor of variance reduction and is obtained from the geostatistical operation of regularisation. It is used to approximate spatial averaging of a correlated variable over some domain by simple averaging of n_e independent variables. Two novel 'parametric' approaches are developed, which honor the frequent requirement of non-negativity and positive skewness and which explore the fact that means of independent gamma and inverse Gaussian variables possess the same distribution type (distribution permanence). The existing method of 'parametric' log-normal up-scaling is included in the study for comparative evaluation against the other methods developed and to define a new parameter for its potential range of applicability. Furthermore, three novel approaches termed 'non-parametric' are presented, which relate to bootstrap sampling and expansions of distributions about the normal (Edgeworth) and gamma distributions. A comprehensive simulation study is performed to validate each method's performance against up-scaling by full random field simulation, and a practical example for up-scaling of core sample rock strength data to drilled shaft side friction resistance with its associated risk of failure is presented.

As expected, results of the simulation study show that the parametric approaches produce acceptable results only when the distribution chosen for approximation is close enough to the actual test distribution. To decide which one (or if any) of the gamma, inverse Gaussian or log-normal is able to give an acceptable result, a simple criterion based on the ratio of the coefficient of skewness to the coefficient of variation of the test distribution is developed and successfully validated. The bootstrap approach is seen to be almost uniformly superior to the others; however, for values of n_e close to 1, previous smoothing of a possibly discontinuous distribution used for the point variable may be necessary to avoid discontinuity artifacts in the up-scaled distribution. Edgeworth and gamma expansions involving skewness and kurtosis are able to correct for a certain degree of deviations in the up-scaled distributions with respect to the normal and gamma distribution, respectively. The parameters Δsk_N and Δsk_{GA} based on skewness and coefficient of variation of the point support variable as well as n_e are developed to find $\Delta sk_N \leq 1$ and $\Delta sk_{GA} \leq 0.5$ as respective ranges of applicability of the Edgeworth and gamma expansion methods. For $\Delta sk_N \leq 0.15$ (and in the absence of extreme situations such as bimodality) the Central Limit Theorem is seen to effectively normalise the up-scaled distributions. Once n_e is determined (typically from numerical integration of the spatial correlation function or graphically as in Klammler *et al.* 2010) all of the approaches presented are simple to apply and honor the Central Limit Theorem for large n_e.

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Appendix 1. Bootstrap sampling for non-integer n_e

Given n_e is an integer it is straightforward to draw n_e independent random numbers from a known discrete or continuous distribution (Efron and Tibshirani 1998). If a large number of N samples of size n_e are drawn, then the respective N arithmetic averages approach a variance of $\sigma_v^2 = \sigma_0^2/n_e$ [Equations (4) and (5)]. Furthermore, if N_1 samples of size n_{e1} and N_2 samples of size n_{e2} are drawn, where $N_1 + N_2 = N$, then the respective N arithmetic averages approach a variance of $\sigma_v^2 = (N_1\sigma_0^2/n_{e1} + N_2\sigma_0^2/n_{e2})/N$. By choosing $n_{e1} = \text{int}(n_e)$, designating the integer part of n_e (e.g. int(3.78) = 3), and $n_{e2} = \text{int}(n_e) + 1$, with $d = n_e - \text{int}(n_e)$ being the non-integer part of n_e, the preceding equations can be combined and written as

$$\frac{1}{n_{e1} + d} = \frac{b_1}{n_{e1}} + \frac{1 - b_1}{n_{e1} + 1} \tag{A1}$$

$b_1 = N_1/N$ and $1 - b_1 = N_2/N$ are used for the portions of samples drawn of size n_{e1} and n_{e2}, respectively. Solving for b_1 gives

$$b_1 = \frac{n_{e1}(1 - d)}{n_{e1} + d} \tag{A2}$$

Hence, the distribution of N sample means of non-integer size n_e is approximated by the requirement of equal variance reduction through the composite distribution of $N_1 = \text{round}(b_1 N)$ samples of integer size n_{e1} and $N_2 = N - N_1$ samples of size $n_{e1} + 1$. The expression 'round()' designates the nearest integer to the argument. Using $n_{e1} = \text{round}(n_e)$ and $N_1 = N$ is an accurate approximation for large n_e (> 100), since $n_{e1}/n_e \approx 1$. For an intermediate range of n_e (≈ 10) Equation (A2) can be simplified to $b_1 = 1 - d$, since $0 \leq d < 1$ and $d/n_e \approx 0$. Equation (A2) is most general and remains accurate even for small values of n_e close to 1.

Supplemental material

Online supplemental material is available with expressions for pdfs and cdfs of distributions used (normal, gamma, inverse Gaussian, log-normal), a detailed description of the simulation study and core sample rock strength data for the practical example.