Acta Math. Hungar., **132 (1–2)** (2011), 168–173
DOI: 10.1007/s10474-010-0047-1
First published online December 11, 2010

GENERALIZED QUOTIENT TOPOLOGIES

L. E. DE ARRUDA SARAIVA

ICAD, Universidade Federal da Bahia, CEP 47.805-100, Barreiras, BA, Brazil
e-mail: lyngnys@ufba.br

(Received June 28, 2010; accepted July 12, 2010)

Abstract. A definition of a generalized quotient topology is given and some characterizations of this concept, up to generalized homeomorphisms, are furnished. For the first approach, we exhibit a monotonic map spanning that generalized quotient topology. We also prove that the notions of generalized normality and generalized compactness are preserved by those quotient structures.

1. Introduction

In [1] Á. Császár introduced and extensively studied the notion of generalized open sets. Since then, he and many other authors have shown that important properties and results still hold, with some or no modification, if the intersection axiom is discarded from the general topology.

Quotient structures play an important role in many geometric and topological constructions. In this paper we generalize the quotient topology in a natural manner and prove some characterizations. Thereafter we give equivalent ways of considering that concept through partitions and, which amounts to be the same, equivalence relations on the underlying set. Finally, we show that the definition preserves, as expected, generalized compactness and normality.

2. Preliminaries

Let X be a nonempty set with $\exp X$ its power set. A class $\mathfrak{g} \subset \exp X$ is called a *generalized topology* (GT for short) on X when $\emptyset \in \mathfrak{g}$ and the union of every family of members of $\mathfrak{g}$ is again a member of $\mathfrak{g}$. Evidently, every topology is a generalized topology. The elements of $\mathfrak{g}$ are called the generalized open sets in X. A pair $(X, \mathfrak{g})$, where $X \neq \emptyset$ is a set and $\mathfrak{g}$ is a GT on it, is said to be a generalized topological space (briefly GTS).

Key words and phrases: generalized quotient topology, decomposition space, natural map.
2000 Mathematics Subject Classification: 54A05, 54B15, 54C10.

By definition, a set $A \subset X$ in a GTS $(X, \mathfrak{g})$ is $\mathfrak{g}$-open iff $A \in \mathfrak{g}$ and $\mathfrak{g}$-closed iff $X - A \in \mathfrak{g}$. When many GTS's are involved, we use the underlying set as an index in $\mathfrak{g}$. Let us denote by $\Gamma(X)$ the class of all *monotonic* maps γ of $\exp X$ into itself, i.e. such that $A \subset B \subset X \Rightarrow \gamma(A) \subset \gamma(B)$. If $\gamma \in \Gamma(X)$, $A \subset X$ is called a γ-open set iff $A \subset \gamma(A)$ and γ-closed iff its complement $X - A$ is γ-open. It can be shown ([1], Proposition 1.1) that the class of all γ-open sets in X constitutes a GT $\mathfrak{g}_\gamma$ on X. Reciprocally, for every GT on X, there is a $\gamma \in \Gamma(X)$ satisfying $\mathfrak{g} = \mathfrak{g}_\gamma$ ([2], Lemma 1.1). Moreover, the intersection of γ-closed sets being γ-closed, it is possible to define the γ-closure of $A \subset X$ as the intersection of all γ-closed sets containing A, denoted by $c_\gamma A$ (see [1]).

According to [2], let us say that a function between GTS's $f : (X, \mathfrak{g}_X) \to (Y, \mathfrak{g}_Y)$ is $(\mathfrak{g}_X, \mathfrak{g}_Y)$-*continuous* if $f^{-1}(V) \in \mathfrak{g}_X$ whenever $V \in \mathfrak{g}_Y$. We shall also say that f is $(\mathfrak{g}_X, \mathfrak{g}_Y)$-*open* ($(\mathfrak{g}_X, \mathfrak{g}_Y)$-*closed*) when the direct image of $\mathfrak{g}_X$-open ($\mathfrak{g}_X$-closed) sets in X is always $\mathfrak{g}_Y$-open ($\mathfrak{g}_Y$-closed) in Y. A $(\mathfrak{g}_X, \mathfrak{g}_Y)$-*homeomorphism* is a $(\mathfrak{g}_X, \mathfrak{g}_Y)$-continuous and $(\mathfrak{g}_X, \mathfrak{g}_Y)$-open function.

A GTS $(X, \mathfrak{g})$ is $\mathfrak{g}$-*compact* (compact in [5]) iff, given an arbitrary cover $\{G_k\}_{k \in K}$ by $\mathfrak{g}$-open sets in X (i.e. $G_k \in \mathfrak{g}$ for each k and $\bigcup_k G_k = X$), it is possible to obtain a nonempty subset $K_0 \subset K$ for which $\bigcup_{k \in K_0} G_k = X$ still holds (see also [3]).

Other important generalization introduced and investigated by Á. Császár is the notion of generalized normality [4].

A GTS $(X, \mathfrak{g})$ is said to be $\mathfrak{g}$-*normal* (normal [5]) when, given disjoint $\mathfrak{g}$-closed sets $F_1, F_2 \subset X$, there are disjoint $\mathfrak{g}$-open sets $G_1, G_2 \subset X$ such that $F_i \subset G_i$.

From now on we write fg for $f \circ g$, whenever composition is possible.

3. Generalized quotient topologies

Given a GTS $(X, \mathfrak{g}_X)$, $Y \neq \emptyset$ a set and $\pi : X \to Y$ a surjective function, it is easy to see that $\mathfrak{g}_\pi = \{ G \subset Y : \pi^{-1}(G) \in \mathfrak{g}_X \}$ is a GT on Y. Indeed, $\emptyset \subset Y$ and $f^{-1}(\emptyset) = \emptyset \in \mathfrak{g}_X$. Moreover, if $\{G_k\}_{k \in K} \subset \mathfrak{g}_\pi$, then $\pi^{-1}(\cup_k G_k) = \cup_k \pi^{-1}(G_k) \in \mathfrak{g}_X$. It will be the object of our first definition:

Definition 3.1. We shall call $\mathfrak{g}_\pi$ the *generalized quotient topology* (briefly GQT) induced on Y by π and $(Y, \mathfrak{g}_\pi)$ will be called the *generalized quotient space* (GQS for short) of X. π is the *generalized quotient map.*

The generalized quotient topology $\mathfrak{g}_\pi$ is the largest GT $\mathfrak{g}_Y$ on Y making π $(\mathfrak{g}_X, \mathfrak{g}_Y)$-continuous. For let $\mathfrak{g}_Y$ be another GT on Y for which π is $(\mathfrak{g}_X, \mathfrak{g}_Y)$-continuous. Then for each $V \in \mathfrak{g}_Y$, $\pi^{-1}(V) \in \mathfrak{g}_X$. Thus $V \in \mathfrak{g}_\pi$.

We also notice that GQT's can be equivalently described by means of $\mathfrak{g}_X$-closed sets: $F \subset Y$ is $\mathfrak{g}_\pi$-closed iff $\pi^{-1}(F)$ is $\mathfrak{g}_X$-closed.

The theorem below gives necessary and sufficient conditions on π to $\mathfrak{g}_Y$ be a GQT.

THEOREM 3.1. *Let* $(X, \mathfrak{g}_X)$ *and* $(Y, \mathfrak{g}_Y)$ *be GTS's and* $\pi : X \to Y$ *be a surjective function. If* π *is* $(\mathfrak{g}_X, \mathfrak{g}_Y)$*-continuous and either* $(\mathfrak{g}_X, \mathfrak{g}_Y)$*-open or* $(\mathfrak{g}_X, \mathfrak{g}_Y)$*-closed, then the GT* $\mathfrak{g}_Y$ *on* Y *coincides with the GQT* $\mathfrak{g}_\pi$*. Conversely, if* $\mathfrak{g}_Y = \mathfrak{g}_\pi$*, then* π *is* $(\mathfrak{g}_X, \mathfrak{g}_Y)$*-continuous.*

PROOF. To begin with, suppose that π is $(\mathfrak{g}_X, \mathfrak{g}_Y)$-continuous and $(\mathfrak{g}_X, \mathfrak{g}_Y)$-open. Since $\mathfrak{g}_\pi$ is the largest GT making π $(\mathfrak{g}_X, \mathfrak{g}_Y)$-continuous, $\mathfrak{g}_Y \subseteqq \mathfrak{g}_\pi$. On the other hand, if $V \in \mathfrak{g}_\pi$, then by definition of $\mathfrak{g}_\pi$, $\pi^{-1}(V) \in \mathfrak{g}_X$. In virtue of $(\mathfrak{g}_X, \mathfrak{g}_Y)$-openness of π, $\pi\big(\pi^{-1}(U)\big) = U$ belongs to $\mathfrak{g}_Y$. Thus $\mathfrak{g}_\pi \subseteqq \mathfrak{g}_Y$. The case π $(\mathfrak{g}_X, \mathfrak{g}_Y)$-closed is analogous. The second part is obvious. $\square$

THEOREM 3.2. *If* Y *is endowed with the GQT induced by a function* π *of* X *onto* Y*, then* $g : (Y, \mathfrak{g}_Y) \to (Z, \mathfrak{g}_Z)$ *is* $(\mathfrak{g}_Y, \mathfrak{g}_Z)$*-continuous iff* $g\pi$ *is* $(\mathfrak{g}_X, \mathfrak{g}_Z)$*-continuous.*

$$\begin{array}{ccc} X & \xrightarrow{\ \pi\ } & Y \\ {\scriptstyle g\pi}\downarrow & \swarrow {\scriptstyle g} & \\ Z & & \end{array}$$

PROOF. *Necessity.* It is clear.

Sufficiency. Suppose that $g\pi$ is $(\mathfrak{g}_X, \mathfrak{g}_Z)$-continuous and let W be a $\mathfrak{g}_Z$-open set. Thus $(g\pi)^{-1}(W) = \pi^{-1}\big(g^{-1}(W)\big) \in \mathfrak{g}_X$. So by definition of the GQT, $g^{-1}(W) \in \mathfrak{g}_Y$. Hence g is $(\mathfrak{g}_Y, \mathfrak{g}_Z)$-continuous. $\square$

For a surjective function $g : X \to X'$, define $g_- : \exp X' \to \exp X$ by $g_-(A') = g^{-1}(A')$ and $g_+ : \exp X \to \exp X'$ by $g_+(A) = g(A)$. Notice that g_- and g_+ satisfy $A' \subset B' \Rightarrow g_-(A') \subset g_-(B')$ and $A \subset B \Rightarrow g_+(A) \subset g_+(B)$, respectively.

For the next lemma, define $\widetilde{\gamma} = \pi_+ \gamma \pi_-$, where $\pi : (X, \mathfrak{g}) \to (\widetilde{X}, \widetilde{\mathfrak{g}})$ is the generalized quotient map.

LEMMA 3.1. *If* $\gamma \in \Gamma(X)$*, then* $\widetilde{\gamma} \in \Gamma(\widetilde{X})$:

$$\begin{array}{ccc} \exp X & \xrightarrow{\ \gamma\ } & \exp X \\ {\scriptstyle \pi_-}\uparrow & & \downarrow{\scriptstyle \pi_+} \\ \exp \widetilde{X} & \xrightarrow{\ \widetilde{\gamma}\ } & \exp \widetilde{X} \end{array}$$

PROOF. The following implications hold trivially:

$$\widetilde{A} \subset \widetilde{B} \subset \widetilde{X} \Rightarrow \pi_-(\widetilde{A}) \subset \pi_-(\widetilde{B}) \Rightarrow \gamma\pi_-(\widetilde{A}) \subset \gamma\pi_-(\widetilde{B})$$

$$\Rightarrow \pi_+\gamma\pi_-(\widetilde{A}) \subset \pi_+\gamma\pi_-(\widetilde{B}). \qquad \square$$

It is possible to explicitly furnish a nice map that yields the GQT $\widetilde{\mathfrak{g}}$ on $\widetilde{X}$.

THEOREM 3.3. *If* $(X, \mathfrak{g}_\gamma)$ *is a GTS, for* $\gamma \in \Gamma(X)$, *then the map* $\widetilde{\gamma} = \pi_+\gamma\pi_- \in \Gamma(\widetilde{X})$ *satisfies* $\mathfrak{g}_{\widetilde{\gamma}} = \widetilde{\mathfrak{g}}$.

PROOF. The definition of the GT on $\widetilde{X}$ assumes the form:

$$\widetilde{U} \in \widetilde{\mathfrak{g}} \Leftrightarrow \pi^{-1}(\widetilde{U}) \in \mathfrak{g}_\gamma \Leftrightarrow \pi^{-1}(\widetilde{U}) \text{ is } \gamma\text{-open}$$

$$\Leftrightarrow \pi^{-1}(\widetilde{U}) \subset \gamma\pi^{-1}(\widetilde{U}) \Leftrightarrow \widetilde{U} = \pi\pi^{-1}(\widetilde{U}) \subset \pi\gamma\pi^{-1}(\widetilde{U}).$$

Therefore, what we have obtained above can be thus stated: $\widetilde{U} \in \widetilde{\mathfrak{g}} \Leftrightarrow \widetilde{U}$ is $\widetilde{\gamma}$-open. In particular, we can write $\widetilde{\mathfrak{g}} = \mathfrak{g}_{\widetilde{\gamma}}$. $\square$

4. Generalized decomposition spaces

A second approach can be accomplished through partitions.

DEFINITION 4.1. Let $(X, \mathfrak{g})$ be a GTS. A *decomposition* on X is a family of disjoint subsets of X whose union equals X. A decomposition $\mathbf{D}$, when provided with the relation

$$(*) \qquad \mathcal{F} \subset \mathbf{D} \Leftrightarrow \bigcup\{F : F \in \mathcal{F}\} \in \mathfrak{g}_X,$$

is called a *generalized decomposition space.*

PROPOSITION 4.1. *The relation* $(*)$ *produces a GT* $\mathfrak{g}_{\mathbf{D}}$ *on* $\mathbf{D}$.

PROOF. Clearly $\emptyset \subset \mathbf{D}$ and $\bigcup \emptyset = \emptyset \in \mathfrak{g}_X$. Moreover, if $\mathcal{F}_k \subset \mathbf{D}$ for all k, we have $\bigcup_k \mathcal{F}_k \in \mathfrak{g}_X$. $\square$

As the members of $\mathbf{D}$ are disjoint and cover X, it is possible to define a surjective map $P : X \to \mathbf{D}$ by setting $P(x) = A$, where $x \in A \in \mathbf{D}$. P is called the *natural map* of X onto $\mathbf{D}$.

LEMMA 4.1. *The natural map* $P : X \to \mathbf{D}$ *is* $(\mathfrak{g}_X, \mathfrak{g}_{\mathbf{D}})$*-continuous.*

PROOF. Pick a $\mathfrak{g}_{\mathbf{D}}$-open set $\mathcal{F} \subset \mathbf{D}$. Then $\bigcup\{F : F \in \mathcal{F}\} \in \mathfrak{g}_X$. But

$$P^{-1}(\mathcal{F}) = P^{-1}\Big(\bigcup_{F \in \mathcal{F}} \{F\}\Big) = \bigcup_{F \in \mathcal{F}} P^{-1}(\{F\}) = \bigcup_{F \in \mathcal{F}} F. \qquad \square$$

P need not be $(\mathfrak{g}_Y, \mathfrak{g}_{\mathbf{D}})$-open, however, as can be seen below.

EXAMPLE 4.1. Let $\emptyset \neq U_1 \subsetneq A$, $\emptyset \neq U_2 \subset B$ and set $X = A \cup B$. If $\mathfrak{g}_X = \{\emptyset, U_1, U_2, U_1 \cup U_2\}$ is the GT on X, then $\mathfrak{g}_{\mathbf{D}} = \big\{\emptyset, \{\emptyset\}, \{U_1\}, \{U_2\}, \{U_1 \cup U_2\}, \{U_1, U_2\}, \{U_1, U_1 \cup U_2\}, \{U_2, U_1 \cup U_2\}\big\}$. Now $P(U_1) = \{A\} \notin \mathfrak{g}_{\mathbf{D}}$. So P is not $(\mathfrak{g}_X, \mathfrak{g}_{\mathbf{D}})$-open.

THEOREM 4.1. *The GT $\mathfrak{g}_{\mathbf{D}}$ on a decomposition space $\mathbf{D}$ of X is the GQT induced by the natural map $P: X \to \mathbf{D}$.*

PROOF. Since $\mathfrak{g}_P$ is the largest GT on $\mathbf{D}$ which makes P $(\mathfrak{g}_X, \mathfrak{g}_{\mathbf{D}})$-continuous, it follows that $\mathfrak{g}_{\mathbf{D}} \subseteqq \mathfrak{g}_P$. Take $\mathcal{F}$ in $\mathfrak{g}_P$. So $\mathcal{F} = \bigcup_k \{\mathcal{F}_k\}$. Thus $P^{-1}(\mathcal{F}) = \bigcup_k P^{-1}\big(\{\mathcal{F}_k\}\big) \in \mathfrak{g}_X$, by Lemma 4.1. Now by definition of P, $P^{-1}\big(\{\mathcal{F}_k\}\big) = \mathcal{F}_k$ for all k. Hence $\mathcal{F} = \bigcup_k \{\mathcal{F}_k\} \in \mathfrak{g}_{\mathbf{D}}$. $\square$

THEOREM 4.2. *If Y is equipped with the GQT induced by a surjective function $\pi: X \to Y$, then Y is $(\mathfrak{g}_Y, \mathfrak{g}_{\mathbf{D}})$-homeomorphic to a decomposition space $\mathbf{D}$, under a $(\mathfrak{g}_Y, \mathfrak{g}_{\mathbf{D}})$-homeomorphism $h: Y \to \mathbf{D}$ such that $h\pi$ is the decomposition map of X onto $\mathbf{D}$.*

PROOF. $\mathbf{D} = \big\{\pi^{-1}(y)\big\}_{y \in Y}$ is a decomposition of X. So define h by $h(y) = \pi^{-1}(y)$. It is obviously bijective. Moreover since π is onto Y, given $x \in X$, there is $y \in Y$ such that $x \in \pi^{-1}(y)$. Thus $\pi(x) = y$ and $(h\pi)(x) = h(y) = \pi^{-1}(y) = P(x)$.

We show that h is $(\mathfrak{g}_Y, \mathfrak{g}_{\mathbf{D}})$-continuous. To show that h is $(\mathfrak{g}_Y, \mathfrak{g}_{\mathbf{D}})$-open is similar. Let $\big\{\pi^{-1}(y)\big\}_{y \in G}$ be a $\mathfrak{g}_{\mathbf{D}}$-open set, for some $G \subset X$. That means $\bigcup_{y \in G} \pi^{-1}(y) \in \mathfrak{g}_X$. On the other hand,

$$\bigcup_{y \in G} \pi^{-1}(y) = \pi^{-1}\Big(\bigcup_{y \in G} \{y\}\Big) = \pi^{-1}(G) \in \mathfrak{g}_X \Leftrightarrow G \in \mathfrak{g}_\pi = \mathfrak{g}_Y.$$

If $h: y \mapsto \pi^{-1}(y)$, then $h^{-1}\big(\pi^{-1}(y)\big) = y$. So

$$h^{-1}\big(\big\{\pi^{-1}(y)\big\}_{y \in G}\big) = h^{-1}\Big(\bigcup_{y \in G} \big\{\pi^{-1}(y)\big\}\Big)$$

$$= \bigcup_{y \in G} h^{-1}\big(\big\{\pi^{-1}(y)\big\}\big) = \bigcup_{y \in G} \{y\} = G.$$

Now $\big\{\pi^{-1}(y)\big\}_{y \in G} \in \mathfrak{g}_{\mathbf{D}} \Rightarrow G \in \mathfrak{g}_\pi = \mathfrak{g}_Y \Rightarrow h$ is $(\mathfrak{g}_Y, \mathfrak{g}_{\mathbf{D}})$-continuous, as desired. $\square$

DEFINITION 4.2. Consider a GTS $(X, \mathfrak{g}_X)$ and an equivalence relation $\sim$ on the underlying set X. The *identification space* $X/\sim$ is defined to be the decomposition space $\mathbf{D}$ the elements of which are the equivalence classes under $\sim$.

REMARK. The importance of Theorems 4.1 and 4.2 is that they ensure that the two constructions are the same, up to generalized homeomorphisms. Now a well-known result from set theory shows that equivalence classes in X partition X, so that this third approach is equivalent to the preceding one, and thus the three definitions introduced are all equivalent.

5. Applications

Let $(X, \mathfrak{g})$ be a GTS and consider the natural map $\pi : X \to \widetilde{X}$; suppose $\widetilde{X}$ is provided with its generalized quotient topology $\widetilde{\mathfrak{g}}$.

THEOREM 5.1. *If X is $\mathfrak{g}$-normal and π is $(\mathfrak{g}, \widetilde{\mathfrak{g}})$-open, then $\widetilde{X}$ is $\widetilde{\mathfrak{g}}$-normal.*

PROOF. Let F, F' be disjoint $\widetilde{\mathfrak{g}}$-closed sets. So $\pi^{-1}(F)$, $\pi^{-1}(\widetilde{F})$ are disjoint $\mathfrak{g}$-closed in X. By hypothesis there are V, V' disjoint $\mathfrak{g}$-open sets such that $\pi^{-1}(F) \subset V$, $\pi^{-1}(F') \subset V'$. Thus $F \subset \pi(V)$ and $F' \subset \pi(V')$. Since π is $(\mathfrak{g}, \widetilde{\mathfrak{g}})$-open, obviously $\pi(V)$, $\pi(V')$ are disjoint $\widetilde{\mathfrak{g}}$-open sets. Therefore $\widetilde{X}$ is $\widetilde{\mathfrak{g}}$-normal. $\square$

THEOREM 5.2. *If X is $\mathfrak{g}$-compact, then $\widetilde{X}$ is $\widetilde{\mathfrak{g}}$-compact.*

PROOF. Let $\widetilde{X} = \cup \widetilde{A}_{k \in K}$ a $\widetilde{\mathfrak{g}}$-open cover of $\widetilde{X}$. So from the $\mathfrak{g}$-open cover $X = \cup \pi^{-1}(\widetilde{A}_k)$ of X, we obtain a finite $\mathfrak{g}$-open subcover $X = \pi^{-1}(\widetilde{A}_{k_1}) \cup \dots \cup \pi^{-1}(\widetilde{A}_{k_n})$. Thus since π is surjective we have $\widetilde{X} = \pi\pi^{-1}(\widetilde{A}_{k_1}) \cup \dots \cup \pi\pi^{-1}(\widetilde{A}_{k_n}) \subset \widetilde{A}_{k_1} \cup \dots \cup \widetilde{A}_{k_n} \subset \widetilde{X}$. Hence $\widetilde{X}$ is $\widetilde{\mathfrak{g}}$-compact. $\square$

References

[1] Á. Császár, Generalized open sets, *Acta Math. Hungar.*, **75** (1997), 65–87.
[2] Á. Császár, Generalized topology, generalized continuity, *Acta Math. Hungar.*, **96** (2002), 351–357.
[3] Á. Császár, γ-compact spaces, *Acta Math. Hungar.*, **87** (2000), 99–107.
[4] Á. Császár, Normal generalized topologies, *Acta Math. Hungar.*, **115** (2007), 309–313.
[5] R. Shen, Remarks on products of generalized topologies, *Acta Math. Hungar.*, **124** (2009), 363–369.